Copula goodness-of-fit testing: An overview and power comparison

Daniel Berg

Department of Mathematics, University of Oslo & The Norwegian Computing Center
PO Box 114, Blindern. NO-0314 Oslo, Norway
E-mail: daniel@danielberg.no

Abstract. Several copula goodness-of-fit approaches are examined, three of which are proposed in this paper. Results are presented from an extensive Monte Carlo study, where we examine the effect of dimension, sample size and strength of dependence on the nominal level and power of the different approaches. While no approach is always the best, some stand out and conclusions and recommendations are made. A novel study of p-value variation due to permuation order, for approaches based on Rosenblatt's transformation is also carried out. Results show significant variation due to permutation order for some of the approaches based on this transform. However, when approaching rejection regions, the additional variation is negligible.

Keywords: Copula, Cramér-von Mises statistic, empirical copula, goodness-of-fit, parametric bootstrap, pseudo-samples, Rosenblatt's transformation

1. Introduction

A copula contains all the information about the dependency structure of a continuous random vector $\mathbf{X} = (X_1, \ldots, X_d)$. Due to the representation theorem of Sklar (1959), every distribution function H can be written as $H(x_1, \ldots, x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\}$, where $F_1, \ldots, F_d$ are the marginal distributions and $C : [0,1]^d \to [0,1]$ is the copula. This enables the modelling of marginal distributions and the dependence structure in separate steps. This feature in particular has motivated successful applications in areas such as survival analysis, hydrology, actuarial science and finance. For exhaustive and general introductions to copulae, the reader is referred to Joe (1997) and Nelsen (1999), and for introductions oriented to financial applications, Malevergne and Sornette (2006) and Cherubini et al. (2004). While the evaluation of univariate distributions is well documented, the study of goodness-of-fit (g-o-f) tests for copulas emerged only recently as a challenging inferential problem.

Let C be the underlying d-variate copula of a population. Suppose one wants to test the composite g-o-f hypothesis

$$\mathcal{H}_0 : C \in \mathcal{C} = \{C_\theta; \theta \in \Theta\} \quad \text{vs.} \quad \mathcal{H}_1 : C \notin \mathcal{C} = \{C_\theta; \theta \in \Theta\}, \tag{1}$$

where Θ is the parameter space. Lately, several contributions have been made to test this hypothesis, e.g. Genest and Rivest (1993), Shih (1998), Breymann et al. (2003), Malevergne and Sornette (2003), Scaillet (2005), Genest and Rémillard (2008), Fermanian (2005), Panchenko (2005), Berg and Bakken (2005), Genest et al. (2006a), Dobrić and Schmid (2007), Quessy et al. (2007), Genest et al. (2008), among others. However, general guidelines and recommendations are sparse.

For univariate distributions, the g-o-f assessment can be performed using e.g. the well-known Anderson-Darling statistic (Anderson and Darling, 1954), or less quantitatively using a QQ-plot. In the multivariate domain there are fewer alternatives. A simple way to build g-o-f approaches for multivariate random variables is to consider multi-dimensional chi-square approaches, as in Dobrić and Schmid (2005) for example. The problem with this approach, as with all binned approaches based on gridding the probability space, is that they will not be feasible for high dimensional problems due to the curse of dimensionality. Another issue with binned approaches is that the grouping of the data is arbitrary and not trivial. Grouping too coarsely destroys valuable information and the ability to contrast distributions becomes very limited. On the other hand, too small groups lead to a highly irregular empirical cumulative distribution function (cdf) due to the limited amount of data. For these reasons, multivariate binned approaches are not considered in this study. Multivariate kernel density estimation (KDE) approaches such as the ones proposed by Fermanian (2005) and Scaillet (2005) are also excluded from this study as they will simply be too computationally exhaustive for high-dimensional problems. The author believes g-o-f to be most useful for high-dimensional problems since copulae are harder to conceptualize in such cases. Moreover, the consequences of poor model choice are often much greater in higher dimensional problems, e.g. risk assessments for high-dimensional financial portfolios.

The class of dimension reduction approaches is a more promising alternative. Dimension reduction approaches reduce the multivariate problem to a univariate problem, and then apply some univariate test, leading to numerically efficient approaches even for high-dimensional problems. These approaches primarily differ in the way the dimension reduction is carried out. For the univariate test it is common to apply standard univariate statistics such as Kolmogorov- or Cramér-von Mises type statistics. Examples include Breymann et al. (2003), Malevergne and Sornette (2003), Genest et al. (2006a), Berg and Bakken (2005), Quessy et al. (2007) and Genest and Rémillard (2008) among others.

This paper is organized as follows. In Section 2 some preliminaries are presented. Section 3 gives an overview of the nine g-o-f approaches considered, including three new ones. In Section 4 results are presented from an extensive Monte Carlo study where we examine the effect of dimension, sample size and strength of dependence on the nominal level and power of the approaches. Several null- and alternative hypothesis copulae are considered. Further, this section also presents results from a novel numerical study of the effect of permutation order for approaches based on Rosenblatt's transform. Finally, Section 5 discusses our findings and makes some recommendations for future research. In addition, detailed testing procedures, leading to proper P-value estimates for all approaches, are given in the appendix.

2. Preliminaries

For copula g-o-f testing one is interested in the fit of the copula alone. Typically, one does not wish to introduce any distributional assumptions for the marginals. Instead the testing is carried out using rank data. Suppose we have n independent samples $\mathbf{x}_1 = (x_{11}, \ldots, x_{1d}), \ldots, \mathbf{x}_n = (x_{n1}, \ldots, x_{nd})$ from the d-dimensional random vector $\mathbf{X}$. The inference is then based on the so-called pseudo-samples $\mathbf{z}_1 = (z_{11}, \ldots, z_{1d}), \ldots, \mathbf{z}_n = (z_{n1} \ldots, z_{nd})$ from the pseudo-vector $\mathbf{Z}$, where

$$\mathbf{z}_j = (z_{j1}, \ldots, z_{jd}) = \left(\frac{R_{j1}}{n+1}, \ldots, \frac{R_{jd}}{n+1} \right), \tag{2}$$

where R_{ji} is the rank of x_{ji} amongst $(x_{1i}, \ldots, x_{ni})$. The denominator $(n+1)$ is used instead of n to avoid numerical problems at the boundaries of $[0,1]^d$. This transformation of each margin through their normalized ranks is often denoted the empirical marginal transformation. Given the independent samples $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, the pseudo-samples $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ can be considered to be samples from the underlying copula C. However, the rank transformation introduces dependence and $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ are no longer independent samples. The practical consequence is the need for parametric bootstrap procedures to obtain reliable P-value estimates. This is treated in more detail in Secion 3.10.

2.1. Rosenblatt's transformation

The Rosenblatt transformation, proposed by Rosenblatt (1952), transforms a set of dependent variables into a set of independent $U[0,1]$ variables, given the multivariate distribution. The transformation is a universally applicable way of creating a set of i.i.d. $U[0,1]$ variables from any set of dependent variables with known distribution. Given a test for multivariate, independent uniformity, the transformation can be used to test the fit of any assumed model.

Definition 2.1 (Rosenblatt's transformation).
Let $\mathbf{Z} = (Z_1, \ldots, Z_d)$ denote a random vector with marginal distributions $F_i(z_i) = P(Z_i \leq z_i)$ and conditional distributions $F_{i|1\ldots i-1}(Z_i \leq z_i | Z_1 = z_1, \ldots, Z_{i-1} = z_{i-1})$ for $i = 1, \ldots, d$. Rosenblatt's

transformation of $\mathbf{Z}$ *is defined as* $\mathcal{R}(\mathbf{Z}) = (\mathcal{R}_1(Z_1), \ldots, \mathcal{R}_d(Z_d))$ *where*

$$\begin{aligned} \mathcal{R}_1(Z_1) &= P(Z_1 \leq z_1) = F_1(z_1), \\ \mathcal{R}_2(Z_2) &= P(Z_2 \leq z_2 | Z_1 = z_1) = F_{2|1}(z_2|z_1), \\ &\vdots \\ \mathcal{R}_d(Z_d) &= P(Z_d \leq x_d | Z_1 = z_1, \ldots, Z_{d-1} = z_{d-1}) = F_{d|1\ldots d-1}(z_d|z_1, \ldots, z_{d-1}). \end{aligned}$$

The random vector $\mathbf{V} = (V_1, \ldots, V_d)$*, where* $V_i = \mathcal{R}_i(Z_i)$*, is now i.i.d.* $U[0,1]^d$*.*

A recent application of this transformation is multivariate g-o-f tests. The Rosenblatt transformation is applied to the samples $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, assuming a multivariate null hypothesis distribution. Under the null hypothesis, the resulting transformed samples $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ should be independent. Hence a test of multivariate independence is carried out. The null hypothesis is typically a parametric copula family. The parameters of this copula family need to be estimated before performing the transformation.

One advantage with Rosenblatt's transformation in a g-o-f setting is that the null- and alternative hypotheses are the same, regardless of the distribution before the transformation. Hong and Li (2005) report Monte Carlo evidence of multivariate tests using transformed variables outperforming tests using the original random variables. Chen et al. (2004) believe that a similar conclusion also applies to g-o-f tests for copulae.

A disadvantage with tests based on Rosenblatt's transformation is the lack of invariance with respect to the permutation of the variables since there are $d!$ possible permutations. However, as long as the permutation is decided randomly, the results will not be influenced in any particular direction. The practical implications of this disadvantage is studied in Section 4.2.

2.2. *Parameter estimation*

Testing the hypothesis in (1) involves the estimation of the copula parameters $\boldsymbol{\theta}$ by some consistent estimator $\widehat{\boldsymbol{\theta}}$. There are mainly two ways of estimating these parameters; the fully parametric method or a semi-parametric method. The fully parametric method, termed the inference functions for marginals (IFM) method (Joe, 1997), relies on the assumption of parametric, univariate marginals. First, the parameters of the marginals are estimated and then each parametric margin is plugged into the copula likelihood which is then maximized with respect to the copula parameters. Since we treat the marginals as nuisance parameters we rather proceed with the pseudo-samples $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ and the semi-parametric method. This method is denoted the pseudo-likelihood (Demarta and McNeil, 2005) or the canonical maximum likelihood (CML) (Romano, 2002) method, and is described in Genest et al. (1995) and in Shih and Louis (1995) in the presence of censorship. Having obtained the pseudo-samples $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ as described in (2), the copula parameters can be estimated using either maximum likelihood (ML) or using the well-known relations to Kendall's tau.

For elliptical copulae in higher dimensions we estimate pairwise Kendall's taus. These are inverted and gives the components of the correlation- and scale matrices for the Gaussian and Student copulae, respectively. For the Student copula one must also estimate the degree-of-freedom parameter. We follow Mashal and Zeevi (2002) and Demarta and McNeil (2005), who propose a two-stage approach in which the scale matrix is first estimated by inversion of Kendall's tau, and then the pseudo-likelihood function is maximized with respect to the degree-of-freedom ν, using the estimate of the scale matrix. For the Archimedean copulae the parameter is estimated by inversion of Kendall's tau. For dimension $d > 2$ we estimate the parameter as the average of the $d(d-1)/2$ pairs of Kendall's taus.

3. Copula goodness-of-fit approaches

The following nine copula g-o-f approaches are examined:

$\mathcal{A}_1$: Based on Rosenblatt's transformation, proposed by Berg and Bakken (2005). This approach includes, as special cases, the approaches proposed by Malevergne and Sornette (2003), Breymann et al. (2003), and the second approach in Chen et al. (2004).

$\mathcal{A}_2$: Based on the the empirical copula and the copula distribution function, proposed by Genest and Rémillard (2008).

$\mathcal{A}_3$: Based on approach $\mathcal{A}_2$ and the Rosenblatt transformation, proposed by Genest et al. (2008).

$\mathcal{A}_4$: Based on the empirical copula and the cdf of the copula function, proposed by Genest and Rivest (1993), Wang and Wells (2000), Savu and Trede (2004) and Genest et al. (2006a).

$\mathcal{A}_5$: Based on Spearman's dependence function, proposed by Quessy et al. (2007).

$\mathcal{A}_6$: A new approach that extends Shih's test (Shih, 1998) for the bivariate Clayton model to arbitrary dimension.

$\mathcal{A}_7$: Based on the inner product between two vectors as a measure of their distance, proposed by Panchenko (2005).

$\mathcal{A}_8$: A new approach based on approach $\mathcal{A}_7$ and the Rosenblatt transformation.

$\mathcal{A}_9$: A new approach based on averages of the approaches above.

Approaches $\mathcal{A}_1$-$\mathcal{A}_5$ are all dimension reduction approaches, while $\mathcal{A}_6$ is a moment-based approach and $\mathcal{A}_7$-$\mathcal{A}_8$ are denoted full multivariate approaches. For all the dimension reduction approaches the study is restricted to the Cramér-von Mises (CvM) statistic for the unviariate test.

3.1. *Approach* $\mathcal{A}_1$

Berg and Bakken (2005) propose a generalization of the approches proposed by Breymann et al. (2003) and Malevergne and Sornette (2003). The approach is based on Rosenblatt's transformation applied to the pseudo-samples $(\mathbf{z}_1, \dots, \mathbf{z}_n)$ from (2), assuming a null hypothesis copula $C_{\widehat{\theta}}$. Under the null hypothesis the resulting samples $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ are samples from the independence copula $C_\perp$ †.

The dimension reduction of approach $\mathcal{A}_1$ is based on the samples $(\mathbf{v}_1, \dots, \mathbf{v}_n)$:

$$W_{1j} = \sum_{i=1}^{d} \Gamma\{v_{ji}; \boldsymbol{\alpha}\}, \quad j = \{1, \dots, n\}$$

where Γ is any weight function used to weight the information in $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ and $\boldsymbol{\alpha}$ is the set of weight parameters. Any weight function may be used, depending on the use and the region of the unit hypercube one wishes to emphasize. Consider for example the special case $\Gamma\{v_{ji}; \boldsymbol{\alpha}\} = \Phi^{-1}(v_{ji})^2$ which corresponds to the approach proposed by Breymann et al. (2003). If the null hypothesis is the Gaussian copula this is also equivalent with the approach proposed by Malevergne and Sornette (2003). Both of the latter studies apply the Anderson-Darling (Anderson and Darling, 1954) statistic. Berg and Bakken (2005) show that the Anderson-Darling statistic with $\Gamma\{v_{ji}; \boldsymbol{\alpha}\} = |v_{ji} - 0.5|$ performs particularly well for testing the Gaussian null hypothesis. Hence, when performing the numerical studies in Section 4.1 the following two special cases of approach $\mathcal{A}_1$ are considered:

$$\mathcal{A}_1^{(a)} : \Gamma\{v_{ji}; \boldsymbol{\alpha}\} = \Phi^{-1}(v_{ji})^2 \quad \text{and} \quad \mathcal{A}_1^{(b)} : \Gamma\{v_{ji}; \boldsymbol{\alpha}\} = |v_{ji} - 0.5|.$$

For approach $\mathcal{A}_1^{(a)}$ it is easy to see that the distribution F_1 of W_{1j} is a χ^2_d distribution for all j†. Hence we can compare W_{1j} directly with the χ^2_d distribution. However, for approach $\mathcal{A}_1^{(b)}$, and in general, the distribution of W_{1j} is not known and one must turn to a double bootstrap procedure (see Section 3.10) to approximate the cdf F_1 under the null hypothesis. The test observator S_1 of approach $\mathcal{A}_1$ is defined as the cdf of $F_1(W_1)$:

$$S_1(w) = P\{F_1(W_1) \le w\}, \qquad w \in [0, 1].$$

†Since we are working with rank data this is only close to, but not exactly true. This issue is discussed in Section 3.10. Until then it is assumed that this holds

Under the null hypothesis $S_1(w) = w$ for all j. The empirical version of the test observator can be computed as

$$\widehat{S}_1(w) = \frac{1}{n+1}\sum_{j=1}^{n} I\{F_1(W_{1j}) \leq w\}.$$

The appropriate version of the CvM statistic is (shown in Appendix B):

$$\begin{aligned}\widehat{T}_1 &= n\int_0^1 \{\widehat{S}_1(w) - S_1(w)\}^2 \,\mathrm{d}S_1(w)\\ &= \frac{n}{3} + \frac{n}{n+1}\sum_{j=1}^{n}\widehat{S}_1\left(\frac{j}{n+1}\right)^2 - \frac{n}{(n+1)^2}\sum_{j=1}^{n}(2j+1)\widehat{S}_1\left(\frac{j}{n+1}\right).\end{aligned}$$

3.2. *Approach* $\mathcal{A}_2$

Genest and Rémillard (2008) propose to use the copula distribution function for the dimension reduction. The approach is based on the empirical copula process, introduced by Deheuvels (1979):

$$\widehat{C}(\mathbf{u}) = \frac{1}{n+1}\sum_{j=1}^{n} I\left\{Z_{j1} \leq u_1, \ldots, Z_{jd} \leq u_d\right\}. \tag{3}$$

where $\mathbf{Z}_j$ is given by (2) and $\mathbf{u} = (u_1, \ldots, u_d) \in [0,1]^d$. The empirical copula is the observed frequency of $P(Z_1 < u_1, \ldots, Z_d < u_d)$. The idea is to compare $\widehat{C}(\mathbf{z})$ with an estimation $C_{\widehat{\theta}}(\mathbf{z})$ of C_θ. This is a very natural approach for copula g-o-f testing considering that most univariate g-o-f tests are based on a distance between empirical- and null hypothesis distribution functions. Genest et al. (2008) state that, given that it is entirely non-parametric, $\widehat{C}$ is the most objective benchmark for testing the copula g-o-f. A CvM statistic for approach $\mathcal{A}_2$ is (Genest et al., 2008):

$$\widehat{T}_2 = n\int_{[0,1]^d} \left\{\widehat{C}(\mathbf{z}) - C_{\widehat{\theta}}(\mathbf{z})\right\}^2 \,\mathrm{d}\widehat{C}(\mathbf{z}) = \sum_{j=1}^{n}\left\{\widehat{C}(\mathbf{z}_j) - C_{\widehat{\theta}}(\mathbf{z}_j)\right\}^2. \tag{4}$$

3.3. *Approach* $\mathcal{A}_3$

Genest et al. (2008) propose to apply approach $\mathcal{A}_2$ to $\mathbf{V} = \mathcal{R}(\mathbf{Z})$. The idea is then to compare $\widehat{C}(\mathbf{v})$ with the independence copula $C_\perp(\mathbf{v})$. A CvM statistic for approach $\mathcal{A}_3$ becomes (Genest et al., 2008):

$$\widehat{T}_3 = n\int_{[0,1]^d} \left\{\widehat{C}(\mathbf{v}) - C_\perp(\mathbf{v})\right\}^2 \,\mathrm{d}\widehat{C}(\mathbf{v}) = \sum_{j=1}^{n}\left\{\widehat{C}(\mathbf{v}_j) - C_\perp(\mathbf{v}_j)\right\}^2.$$

3.4. *Approach* $\mathcal{A}_4$

Genest and Rivest (1993), Wang and Wells (2000), Savu and Trede (2004) and Genest et al. (2006a) propose to use Kendall's dependence function $K(w) = P(C(\mathbf{Z}) \leq w)$ as a g-o-f approach. The test observator S_4 of approach $\mathcal{A}_4$ becomes

$$S_4(w) = P\{C(\mathbf{Z}\} \leq w\}, \qquad w \in [0,1],$$

where $\mathbf{Z}$ is the pseudo-vector. Under the null hypothesis, $S_4(w) = S_{4,\widehat{\theta}}(w)$ which is copula specific. The empirical version of test observator S_4 equals

$$\widehat{S}_4(w) = \frac{1}{n+1}\sum_{j=1}^{n} I\{\widehat{C}(\mathbf{z}_j) \leq w\}.$$

A CvM statistic for approach $\mathcal{A}_4$ is given by:

$$\widehat{T}_4 = n\int_0^1 \{\widehat{S}_4(w) - S_{4,\widehat{\theta}}(w)\}^2 \,\mathrm{d}\widehat{S}_4(w) = \sum_{j=1}^{n}\left\{\widehat{S}_4\left(\frac{j}{n+1}\right) - S_{4,\widehat{\theta}}\left(\frac{j}{n+1}\right)\right\}^2.$$

3.5. Approach $\mathcal{A}_5$

Quessy et al. (2007) propose a g-o-f approach for bivariate copulae based on Spearman's dependence function $L_2(w) = P(Z_1 Z_2 \leq w)$. Notice that $L_2(w) = P(C_\perp(Z_1, Z_2) \leq w)$. A natural extension to arbitrary dimension d is then $L_d(w) = P(C_\perp(\mathbf{Z}) \leq w)$ and the test observator S_5 of approach $\mathcal{A}_5$ becomes

$$S_5(w) = P\{C_\perp(\mathbf{Z}) \leq w\}, \quad w \in [0,1],$$

where $\mathbf{Z}$ is the pseudo-vector. Under the null hypothesis, $S_5(w) = S_{5,\widehat{\theta}}(w)$, which is copula specific. The empirical version of test observator S_5 equals

$$\widehat{S}_5(w) = \frac{1}{n+1} \sum_{j=1}^{n} I\{C_\perp(\mathbf{z}_j) \leq w\}.$$

A CvM statistic for approach A_5 is given by:

$$\widehat{T}_5 = n \int_0^1 \{\widehat{S}_5(w) - S_{5,\widehat{\theta}}(w)\}^2 \, \mathrm{d}\widehat{S}_5(w) = \sum_{j=1}^{n} \left\{ \widehat{S}_5\left(\frac{j}{n+1}\right) - S_{5,\widehat{\theta}}\left(\frac{j}{n+1}\right) \right\}^2.$$

3.6. Approach $\mathcal{A}_6$

Shih (1998) propose a moment-based g-o-f test for the bivariate gamma frailty model, also known as Clayton's copula. Shih (1998) considered unweighted and weighted estimators of the dependency parameter θ via Kendall's tau and a weighted rank-based estimator, namely

$$\widehat{\theta}_\tau = \frac{2\widehat{\tau}}{1-\widehat{\tau}} \quad \text{and} \quad \widehat{\theta}_W = \frac{\sum_{i<j} \Delta_{ij}/W_{ij}}{\sum_{i<j}(1-\Delta_{ij})/W_{ij}},$$

where $\widehat{\tau} = -1 + 4\sum_{i<j} \Delta_{ij}/\{n(n-1)\}$, $\Delta_{ij} = I\{(Z_{i1} - Z_{j1})(Z_{i2} - Z_{j2}) > 0\}$ and $W_{ij} = \sum_{k=1}^{n} I\{Z_{k1} \leq \max(Z_{i1}, Z_{j1}), Z_{k2} \leq \max(Z_{i2}, Z_{j2})\}$. Since $\widehat{\theta}_\tau$ and $\widehat{\theta}_W$ are both unbiased estimators of θ under the null hypothesis that $C = C_\theta$ for some $\theta \geq 0$, Shih (1998) propose the g-o-f statistic

$$\widehat{T}_{Shih} = \sqrt{n}\{\widehat{\theta}_\tau - \widehat{\theta}_W\}.$$

Shih (1998) shows that this statistic is asymptotically normal under the null hypothesis. Unfortunately, the variance provided by Shih (1998) was found to be wrong by Genest et al. (2006b), where a corrected formula is provided.

One way of extending this approach to arbitrary dimension d is comparing each pairwise element of $\widehat{\theta}_\tau$ and $\widehat{\theta}_W$. The resulting vector of $d(d-1)/2$ statistics will tend, asymptotically, to a $d(d-1)/2$ dimensional normal vector with a non-trivial covariance matrix. The normalized version of the vector, i.e. the inverted square root of the covariance matrix multiplied with the vector of statistics, will be asymptotically standard normal and hence the sum of squares will now be chi-squared with $d(d-1)/2$ degrees of freedom. The covariance matrix of the vector of statistics remains to be computed and is deferred to future research. For now we simply compute the non-normalized sum of squares and perform a parametric bootstrap (see Section 3.10) to estimate the P-value.

The test statistic for approach $\mathcal{A}_6$ then becomes:

$$\widehat{T}_6 = \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \left\{ \widehat{\theta}_{\tau,ij} - \widehat{\theta}_{W,ij} \right\}^2.$$

$\widehat{\theta}_W$, and hence approach $\mathcal{A}_6$, is constructed specifically for testing the Clayton copula and will not be considered for testing any other copula model.

3.7. *Approach* $\mathcal{A}_7$

Panchenko (2005) propose to test the entire data set in one step. The approach is based on the inner product of $\mathbf{Z}$ and $\mathbf{Z}_{\widehat{\theta}}$, where $\mathbf{Z}$ is the pseudo-vector and $\mathbf{Z}_{\widehat{\theta}}$ is the null hypothesis vector with $\widehat{\theta}$ being a consistent estimator of the copula parameter. The inner product can be used as a measure of the distance between two vectors. Now define the squared distance Q between the two vectors as

$$Q = \left\langle \mathbf{Z} - \mathbf{Z}_{\widehat{\theta}} \; |\kappa_d| \; \mathbf{Z} - \mathbf{Z}_{\widehat{\theta}} \right\rangle .$$

Here κ_d is a positive definite symmetric kernel such as the Gaussian kernel:

$$\kappa_d(\mathbf{Z}, \mathbf{Z}') = \exp\left\{ -\|\mathbf{Z} - \mathbf{Z}'\|^2 / (2dh^2) \right\},$$

with $\|\cdot\|$ denoting the Euclidean norm in $\mathbb{R}^d$ and $h > 0$ being a bandwidth. Q will be zero if and only if $\mathbf{Z} = \mathbf{Z}_{\widehat{\theta}}$. Suppose we have the random samples $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ from $\mathbf{Z}$. Now generate the random samples $(\mathbf{z}_1^*, \ldots, \mathbf{z}_n^*)$ from the null hypothesis vector $\mathbf{Z}_{\widehat{\theta}}$. Following the properties of an inner product, Q can be decomposed as $Q = Q_{11} - 2Q_{12} + Q_{22}$. Each term of this decomposition is estimated using V-statistics (see Denker and Keller (1983) for an introduction to U- and V-statistics). The test statistic for approach $\mathcal{A}_7$ is given by:

$$\widehat{T}_7 = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_d(\mathbf{z}_i, \mathbf{z}_j) - \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_d(\mathbf{z}_i, \mathbf{z}_j^*) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_d(\mathbf{z}_i^*, \mathbf{z}_j^*).$$

3.8. *Approach* $\mathcal{A}_8$

Along the lines of approach $\mathcal{A}_3$ we propose a version of approach $\mathcal{A}_7$ based on $\mathbf{V} = \mathcal{R}(\mathbf{Z})$. Given the random samples $(\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*)$, drawn from the independence copula, the statistic for approach $\mathcal{A}_8$ is simply

$$\widehat{T}_8 = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_d(\mathbf{v}_i, \mathbf{v}_j) - \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_d(\mathbf{v}_i, \mathbf{v}_j^*) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_d(\mathbf{v}_i^*, \mathbf{v}_j^*).$$

For approaches $\mathcal{A}_7$ and $\mathcal{A}_8$ it may seem odd to base the deviance measure on one single sample from the null hypothesis and that an average over several repetitions would be more accurate. However, $\mathcal{A}_7$ is the approach in Panchenko (2005) and we include all approaches in their unaltered form. For approach $\mathcal{A}_8$ we wish to examine the effect of Rosenblatt's transformation on approach $\mathcal{A}_7$ so we stick to the deviance from one single sample.

3.9. *Approach* $\mathcal{A}_9$

One can imagine that the different approaches capture deviations from the null hypothesis in different ways. Hence, we propose to average several approaches in an attempt to capture these differences. The different approaches are not on the same scale, hence such averages should be taken over standardized variables, i.e. all approaches should be scaled appropriately. However, we include these averages in their simplest, non-standardized form, as suggestions for future research. Two specific averages are considered. First the average of all nine approaches and second the average of three approaches based on the empirical copula, namely $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$. The corresponding statistics are defined as

$$\widehat{T}_9^{(a)} = \frac{1}{9} \left\{ \widehat{T}_1^{(a)} + \widehat{T}_1^{(b)} + \sum_{j=2}^{8} \widehat{T}_j \right\} \quad \text{and} \quad \widehat{T}_9^{(b)} = \frac{1}{3} \left\{ \widehat{T}_2 + \widehat{T}_3 + \widehat{T}_4 \right\}.$$

3.10. *Testing procedure*

In Section 3.1 it was assumed that $\mathbf{V} = \mathcal{R}(\mathbf{Z})$ is i.i.d. $U[0,1]^d$. The use of ranks in the transformation of the marginals introduce sample dependence in $\mathbf{V}$. Thus $\mathbf{V}$ is only close to, but not exactly i.i.d. The consequence is that approximations of the limiting distributions of test statistics are inaccurate. In addition, the distributions depend on the value of the dependence parameter θ. Nevertheless, we can obtain reliable P-value estimates through a parametric bootstrap procedure. The parametric bootstrap procedure used in Genest et al. (2006a) is adopted, the validity of which is established in Genest and Rémillard (2008). The asymptotic validity of the bootstrap procedure has only been proved so far for the approaches $\mathcal{A}_2$ and $\mathcal{A}_4$. However, results herein and in Berg and Bakken (2005); Dobrić and Schmid (2007) strongly indicate validity also for the other approaches.

Detailed test procedures for all approaches can be found in Berg (2007). Here, we restrict the presentation to approach $\mathcal{A}_2$:

(1) Extract the pseudo-samples $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ by converting the sample data $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ into normalized ranks according to (2).

(2) Estimate the parameters θ with a consistent estimator $\widehat{\theta} = \widehat{\mathcal{V}}(\mathbf{z}_1, \ldots, \mathbf{z}_n)$.

(3) Compute $\widehat{C}(\mathbf{z})$ according to (3).

(4) If there is an analytical expression for C_θ, compute the estimated statistic $\widehat{T}_2$ by plugging $\widehat{C}(\mathbf{z})$ and $C_{\widehat{\theta}}(\mathbf{z})$ into (4). Jump to step (5).
If there is no analytical expression for C_θ then choose $N_b \geq n$ and carry out the following steps (double bootstrap):

   (i) Generate a random sample $(\mathbf{x}_1^*, \ldots, \mathbf{x}_{N_b}^*)$ from the null hypothesis copula $C_{\widehat{\theta}}$ and compute the associated pseudo-samples $(\mathbf{z}_1^*, \ldots, \mathbf{z}_{N_b}^*)$ according to (2).

   (ii) Approximate $C_{\widehat{\theta}}$ by $C_{\widehat{\theta}}^*(\mathbf{u}) = \frac{1}{N_b+1}\sum_{l=1}^{N_b} I\{\mathbf{z}_l^* \leq \mathbf{u}\}, \quad \mathbf{u} \in [0,1]^d$.

   (iii) Approximate the CvM statistic in (4) by $\widehat{T}_2 = \sum_{j=1}^{n}\left\{\widehat{C}(\mathbf{z}_j) - C_{\widehat{\theta}}^*(\mathbf{z}_j)\right\}^2$.

(5) For some large integer K, repeat the following steps for every $k \in \{1, \ldots, K\}$ (parametric bootstrap):

   (a) Generate a random sample $(\mathbf{x}_{1,k}^0, \ldots, \mathbf{x}_{n,k}^0)$ from the null hypothesis copula $C_{\widehat{\theta}}$ and compute the associated pseudo-samples $(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$ according to (2).

   (b) Estimate the parameters θ^0 with a consistent estimator $\widehat{\theta}_k^0 = \widehat{\mathcal{V}}(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$.

   (c) Let $\widehat{C}_k^0(\mathbf{u}) = \frac{1}{n+1}\sum_{j=1}^{n} I\{\mathbf{z}_{j,k}^0 \leq \mathbf{u}\}, \quad \mathbf{u} \in [0,1]^d$.

   (d) If there is an analytical expression for C_θ, let $\widehat{T}_{2,k}^0 = \sum_{j=1}^{n}\left\{\widehat{C}_k^0(\mathbf{z}_{j,k}^0) - C_{\widehat{\theta}_k^0}(\mathbf{z}_{j,k}^0)\right\}^2$ and jump to step (6).
   If there is no analytical expression for C_θ then choose $N_b \geq n$ and proceed as follows:

      (i) Generate a random sample $(\mathbf{x}_{1,k}^{0*}, \ldots, \mathbf{x}_{N_b,k}^{0*})$ from the null hypothesis copula $C_{\widehat{\theta}_k^0}$ and compute the associated pseudo-samples $(\mathbf{z}_{1,k}^{0*}, \ldots, \mathbf{z}_{N_b,k}^{0*})$ according to (2).

      (ii) Approximate $C_{\widehat{\theta}_k^0}$ by $C_{\widehat{\theta}_k^0}^{0*}(\mathbf{u}) = \frac{1}{N_b+1}\sum_{l=1}^{N_b} I\{\mathbf{z}_{l,k}^{0*} \leq \mathbf{u}\}, \quad \mathbf{u} \in [0,1]^d$,

      (iii) Approximate the CvM statistic in (4) by $\widehat{T}_{2,k}^* = \sum_{j=1}^{n}\left\{\widehat{C}_k^0(\mathbf{z}_{j,k}^0) - C_{\widehat{\theta}_k^0}^{0*}(\mathbf{z}_{j,k}^0)\right\}^2$.

(6) An approximate P-value for approach $\mathcal{A}_2$ is then given by $\widehat{p} = \frac{1}{K+1}\sum_{k=1}^{K} I\{\widehat{T}_{2,k}^0 > \widehat{T}_2\}$.

In this parametric bootstrap procedure there are two parameters that needs to be chosen, the sample size N_b for the double bootstrap step (step 4) and the number of replications K (step 5) for the estimation of P-values. In this paper the number of replications $K = 1000$ while the double bootstrap sample size $N_b = 10000$ for approach $\mathcal{A}_1$, and for approaches $\mathcal{A}_2$, $\mathcal{A}_4$ and $\mathcal{A}_5$ $N_b = 2500$ for dimensions $d = \{2, 4\}$ and $N_b = 5000$ for dimension $d = 8$. See Berg (2007) for details.

4. Numerical experiments

4.1. Size and power simulations

A large Monte Carlo study is performed to assess the properties of the approaches for various dimensions, sample sizes, levels of dependence and alternative dependence structures. The nominal levels and the power against fixed alternatives are of particular interest. The simulations are carried out according to the following factors:

- $\mathcal{H}_0$ copula (5 choices: Gaussian, Student, Clayton, Gumbel, Frank),
- $\mathcal{H}_1$ copula (5 choices: Gaussian, Student ($\nu = 6$), Clayton, Gumbel, Frank),
- Kendall's tau (2 choices: $\tau = \{0.2, 0.4\}$),
- Dimension (3 choices: $d = \{2, 4, 8\}$),
- Sample size (2 choices: $n = \{100, 500\}$).

Due to extreme computational load, the Student copula is only considered as null hypothesis in the bivariate case. In each of the remaining 240 cases, a sample of dimension d and size n is drawn from the $\mathcal{H}_1$ copula with dependence parameter corresponding to τ. The statistics of the various g-o-f approaches are then computed under the null hypothesis $\mathcal{H}_0$ and P-values are estimated. This entire procedure is repeated 10000 times in order to estimate the nominal level and power for each approach under consideration.

Since we apply a parametric bootstrap procedure in the estimation of P-values, critical values are obtained by simulating from the null hypothesis, and hence all reported powers are so-called size-adjusted powers and approaches can be compared appropriately (see e.g. Hendry (2006) and Florax et al. (2006) for size-adjustment suggestions).

A natural way of comparing approaches would be to rank their performance. However, an approach can be almost as good as the best approach in all cases but not necessarily the very best. For example when testing the Gaussian copula where the alternative is the Gumbel copula for $d = 4, n = 500$ and $\tau = 0.40$, approach $\mathcal{A}_9^2$ will be ranked 1 with a power of 99.8 while approach $\mathcal{A}_5$ will be ranked number 5 with a power of 98.1. This small difference may not be statistically significant and purely due to Monte Carlo variation. Hence, we rather consider boxplots showing the differences in power from the best performing approach. We also present average powers for combinations of dimension and sample size, i.e. averaged over dependency levels and alternative copulae.

Sections 4.1.1-4.1.5 presents power difference boxplots and average power tables for testing the Gaussian, Student, Clayton, Gumbel and Frank null hypotheses. Detailed tables with all power results are deferred to Appendix C.

The critical values of each statistic under the true null hypothesis are tabulated for each dimension and sample size and for many levels of dependence. For the power simulations we used table look-up with linear interpolation to ensure comparison with the appropriate critical value. Despite the tabulation this computationally exhaustive experiment would not have been feasible without access to the *Titan* computer grid at the University of Oslo, a cluster of (at the time) $1,750$ computing cores, 6.5 TB memory, 350 TB local disk and 12.5 Tflops.

4.1.1. Testing the Gaussian hypothesis

Let us first consider testing the Gaussian hypothesis under several fixed alternatives. The following summary can be read from Figure 1, Table 1 and the extensive results in Table C.4.

- Nominal levels of all approaches match prescribed size of 5%
- Power generally (but not always) increases with level of dependence.
- Power increases with sample size as it should for the approaches to be consistent

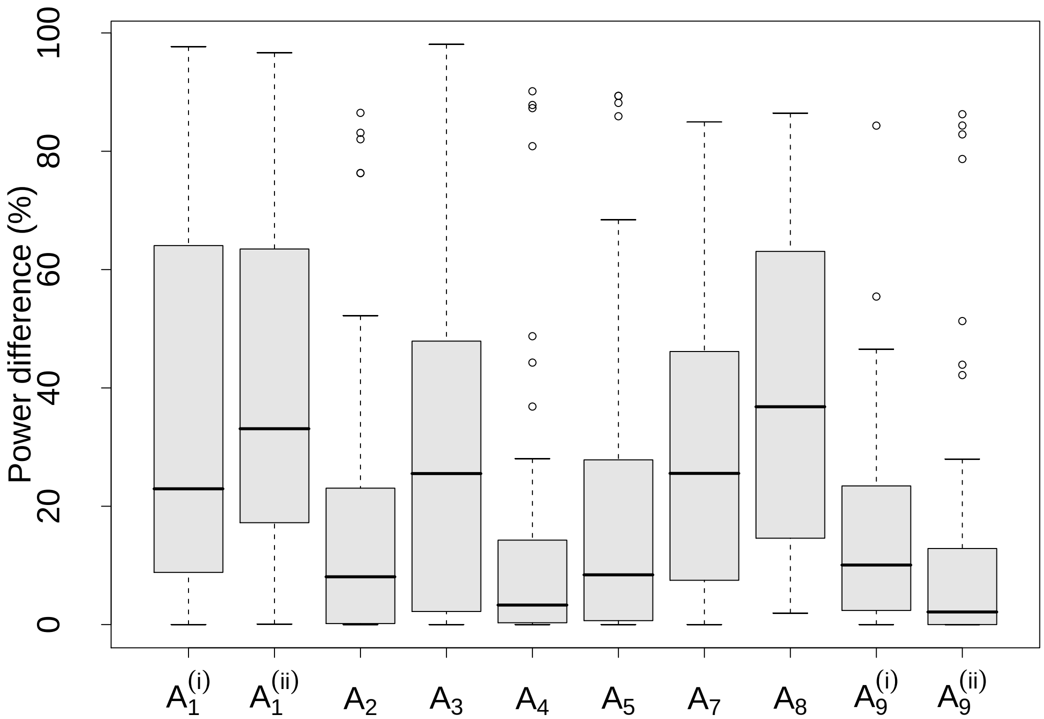

Figure 1. Power differences from the best approach for testing the Gaussian copula.

- Power generally (but not always) increases with dimension, as expected. See e.g. Chen et al. (2004) who show that the Kullbach-Leibler Information Criterion (a measure of distance between two copulae) between the Gaussian- and Student copulae increases with dimension.
- Approaches $\mathcal{A}_4$ and $\mathcal{A}_9^{(b)}$ perform very well and are recommended. However, there are exceptions and additions worth noting:
  - $\mathcal{A}_1$ and $\mathcal{A}_3$ perform particularly well for testing against heavy tails, i.e. the Student copula alternative.
  - $\mathcal{A}_2$ also perform very well for testing against Archimedean alternatives
  - $\mathcal{A}_3$ performs particularly well for the Frank alternative in the bivariate case but very poor for higher dimensions. This illustrates the danger of concluding for higher dimensions based on bivariate results.

4.1.2. Testing the Student hypothesis

Next, we consider testing the Student copula hypothesis, for the bivariate case only. From Figure 2, Table 1 and the extensive results in Table C.5, we can summarize:

- Nominal levels match prescribed size of 5%.
- Powers against Gaussian copula also match prescribed size. This is due to the Gaussian copula being a special case of the Student copula. The statistics are computed by estimating the parameters of the Student copula from the data and hence the Student copula null hypothesis will include the Gaussian copula alternative through a large estimated value for the degree-of-freedom parameter.
- Approaches $\mathcal{A}_2$, $\mathcal{A}_4$ and in particular $\mathcal{A}_9^{(b)}$ perform very well and are recommended.

Table 1. Summary of rejection percentages (at 5% significance level). The results are averaged over sample size, dependency levels and alternative copulae.

$\mathcal{H}_0$	d	$\mathcal{A}_1^{(a)}$	$\mathcal{A}_1^{(b)}$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$	$\mathcal{A}_8$	$\mathcal{A}_9^{(a)}$	$\mathcal{A}_9^{(b)}$
Gauss	2	5.7	5.7	24.7	23.8	23.7	19.1	–	13.1	14.0	18.8	**26.6**
	4	22.1	16.3	37.4	32.1	43.4	39.5	–	27.3	20.8	42.7	**43.9**
	8	34.0	29.8	47.0	27.7	50.3	46.9	–	40.7	24.1	**52.2**	50.1
Student	2	5.2	5.2	23.5	17.1	23.9	19.5	–	12.3	12.1	20.6	**25.0**
Clayton	2	28.7	27.5	57.9	37.9	56.9	46.9	**58.4**	31.0	29.5	57.1	57.4
	4	54.6	42.2	69.0	32.4	70.9	69.2	71.8	46.3	34.2	**73.7**	71.1
	8	63.2	52.5	68.1	37.6	69.8	74.8	77.8	52.5	32.1	**78.3**	70.3
Gumbel	2	14.5	11.8	41.6	30.8	36.8	32.9	–	20.0	19.1	36.3	**39.4**
	4	39.2	35.7	65.7	57.2	65.6	59.0	–	46.4	23.0	64.9	**67.2**
	8	48.5	50.4	72.3	60.6	74.1	62.4	–	62.6	20.8	69.2	**74.6**
Frank	2	11.6	9.1	**33.9**	25.6	31.5	25.9	–	15.2	15.6	30.8	**33.9**
	4	23.9	25.7	58.6	50.1	58.2	51.6	–	32.6	22.3	59.9	**61.0**
	8	36.6	42.8	73.0	67.5	71.0	60.1	–	51.2	24.9	69.9	**73.3**

Note: Numbers in **bold** indicate the best performing approach.

- $\mathcal{A}_1^{(a)}$, $\mathcal{A}_1^{(b)}$, $\mathcal{A}_7$ and $\mathcal{A}_8$ all perform rather poorly.
- For testing the Gaussian and Student hypotheses, powers are in general, as seen from Table 1, lower than for testing the Clayton, Gumbel and Frank hypotheses. This means that it is more difficult to test the elliptical than the Archimedean hypotheses.

4.1.3. Testing the Clayton hypothesis

Figure 3, Table 1 and the extensive results in Table C.6 show the results of testing the Clayton copula hypothesis. We summarize:

- Nominal levels match prescribed size of 5%.
- Approaches $\mathcal{A}_2$, $\mathcal{A}_4$, $\mathcal{A}_9^{(b)}$ and in particular $\mathcal{A}_6$ perform very well and are recommended. $\mathcal{A}_9^{(a)}$ also performs very well but this is largely due to the good performance of $\mathcal{A}_6$ which dominates this average approach since its scale is much larger than the other approaches included in the average.
- $\mathcal{A}_7$, $\mathcal{A}_8$ and in particular $\mathcal{A}_3$ perform very poorly.
- $\mathcal{A}_1^{(a)}$ and $\mathcal{A}_1^{(b)}$ also perform rather poorly.
- Powers are higher than for testing the Gaussian, Student and as we will soon see, the Gumbel and Frank hypotheses, i.e. it is easier to test the Clayton hypothesis.

4.1.4. Testing the Gumbel hypothesis

We now test the Gumbel hypothesis. We summarize from Figure 4, Table 1 and the extensive results in Table C.7:

- Nominal levels match prescribed size of 5%.
- Approaches $\mathcal{A}_2$, $\mathcal{A}_4$ and in particular $\mathcal{A}_9^{(b)}$ perform very well and are recommended.
- $\mathcal{A}_1^{(a)}$, $\mathcal{A}_1^{(b)}$ and in particular $\mathcal{A}_8$ perform very poorly.
- Powers are lower than for testing the Clayton hypothesis but higher than for testing the Gaussian, Student and as we will soon see, the Frank hypotheses.

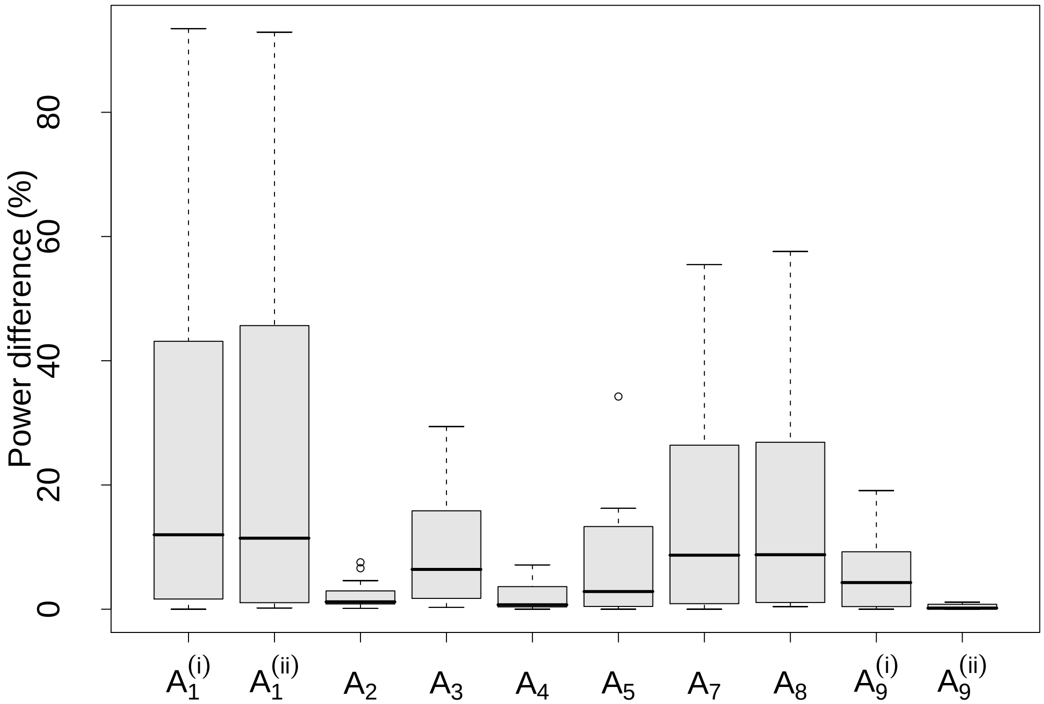

Figure 2. Distribution of power difference from the very best approach for testing the Student copula.

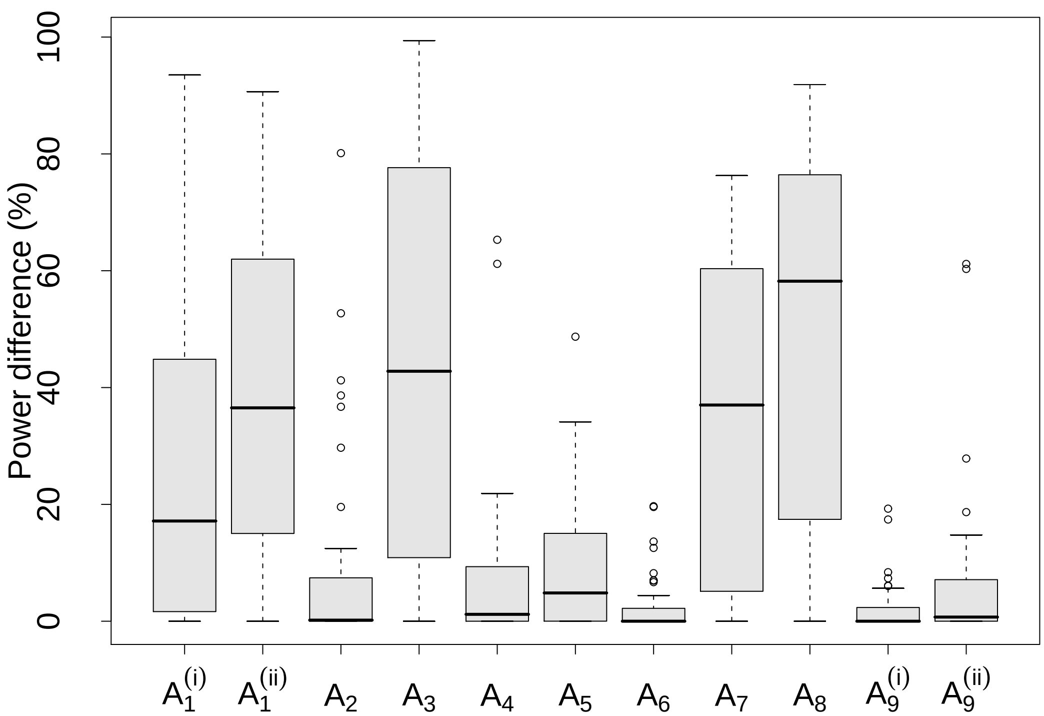

Figure 3. Distribution of power difference from the very best approach for testing the Clayton copula.

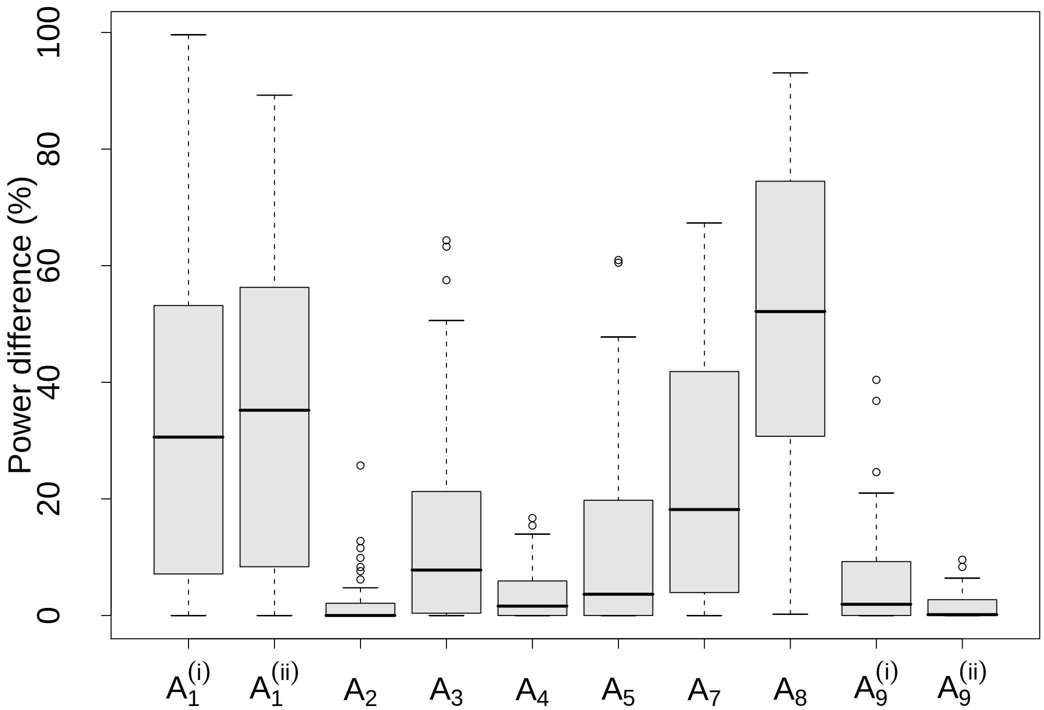

Figure 4. Distribution of power difference from the very best approach for testing the Gumbel copula.

4.1.5. Testing the Frank hypothesis

Finally, we test the Frank hypothesis. From Figure 5, Table 1 and the extensive results in Table C.8 we summarize:

- Nominal levels match prescribed size of 5%.
- Approaches $\mathcal{A}_2$ and in particular $\mathcal{A}_9^{(b)}$ perform very well and are recommended.
- $\mathcal{A}_4$ and $\mathcal{A}_9^{(a)}$ also perform quite well.
- $\mathcal{A}_1^{(a)}$, $\mathcal{A}_1^{(b)}$ and in particular $\mathcal{A}_8$ perform very poorly.
- Powers are higher than for testing the Gaussian and Student hypotheses but lower than for testing the Clayton and Gumbel hypotheses.

4.2. Effect of permutation order for Rosenblatt's transform

Approaches $\mathcal{A}_1$, $\mathcal{A}_3$ and $\mathcal{A}_8$ are all based on Rosenblatt's transform and a consecutive test of independence. The lack of invariance to the order of permutation may pose a problem to these approaches. The statistic for a given data set may prove very different depending on the permutation order. This is an undesirable feature of a statistical testing procedure. However, the practical consequence of this permutation invariance has not yet been investigated.

To examine this effect we draw random samples from an alternative copula $\mathcal{H}_1$. We then compute a P-value, assuming a null copula $\mathcal{H}_0$. This is done for each approach and for each permutation of the variables. We then look at the mean and standard deviation over all permutations. We repeat this procedure 1000 times and report average values in Table 2. The study is restricted to dimension $d = 5$ for which there are $d! = 120$ different permutations, sample size $n = 100$ and dependence $\tau = 0.5$.

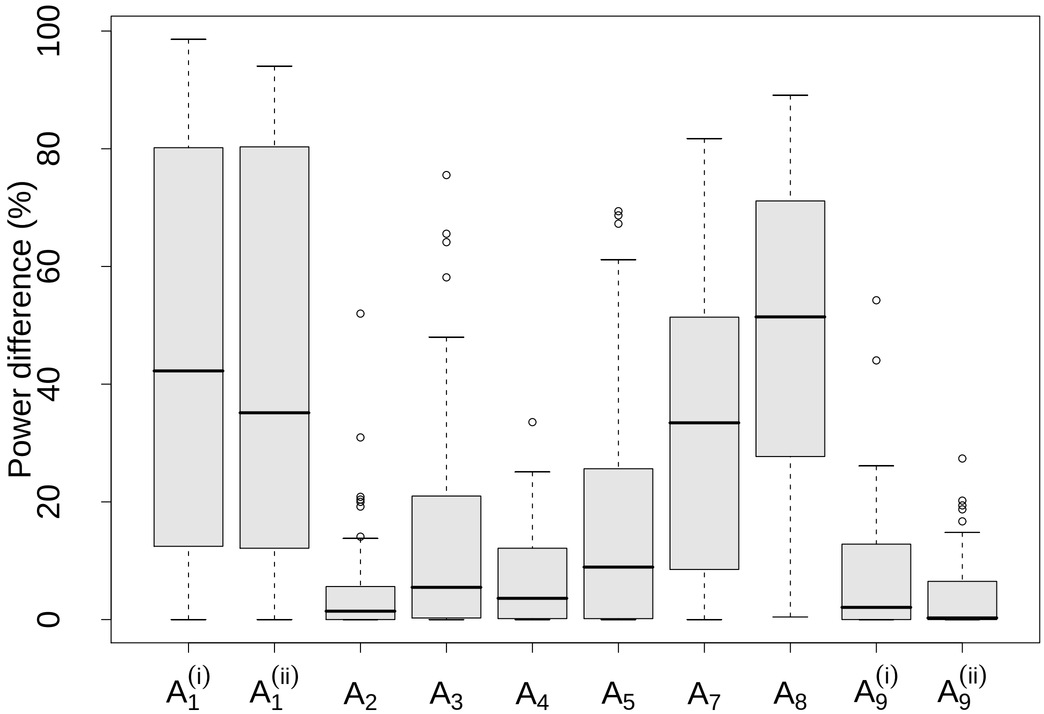

Figure 5. Distribution of power difference from the very best approach for testing the Frank copula.

For some of the approaches there are two sources of variation; permutation order and double bootstrap procedure (see Section 3.10). In order to see the effect of permutation order only, we report the same P-value variation results when the permutation is kept fixed, see Table 3.

From the two tables one can see that the permutation order adds no variance for approach $\mathcal{A}_1^{(a)}$ when the null hypothesis is the Gaussian copula. This permutation invariance of approach $\mathcal{A}_1^{(a)}$ under the Gaussian null hypothesis is proved in Appendix A. However, when using a different weight function or when the null hypothesis is different from the Gaussian copula, variation is added due to the permutation order. Note that in- or close to rejection regions, i.e. in cases where an approach has high power and the P-value is very small, the variation due to permutation order will not have a practical consequence as the conclusion will most probably be rejection of $\mathcal{H}_0$, regardless of permutation order. We see the same for the other approaches. For approach $\mathcal{A}_1^{(a)}$ we see that the variation is in general lower than for the other approaches. Also note that for approach $\mathcal{A}_8$ the permutation order adds almost no variation in any case as the estimated P-value will vary heavily even when keeping the permutation order fixed. This is due to the construction of the approach where random samples from the null hypothesis copula are drawn in every computation of the statistic, inducing large variation.

5. Discussion and recommendations

An overview of six copula g-o-f approaches was given, along with the proposal of three new approaches. A large Monte Carlo study was presented, examining the nominal levels and the power against some fixed alternatives under several combinations of problem dimension, sample size and dependence. Finally we investigated what effect the permutation order has in the Rosenblatt transformation.

Sections 4.1.1-4.1.5 summarize the findings of the Monte Carlo study and provides recommendations to which approach to use in each case. In general we observe increasing power with dimension, sample size and dependence. While no approach strictly dominates the others in terms of power, approaches $\mathcal{A}_2$, $\mathcal{A}_4$ and in particular approach $\mathcal{A}_9^{(b)}$ perform very well, the latter being the overall best performing approach. However, when testing the Gaussian hypothesis against heavy tails, the otherwise poor approach $\mathcal{A}_1$

Table 2. Estimated mean P-values (mean of $d!$ permutations) for approaches based on Rosenblatt's transformation. In parentheses the standard deviation over all permutations. All quoted values are averaged over 1000 simulations.

$\mathcal{H}_0$	$\mathcal{H}_1$	$\mathcal{A}_1^{(a)}$	$\mathcal{A}_1^{(b)}$	$\mathcal{A}_3$	$\mathcal{A}_8$
Gaussian	Gaussian	0.514 (0.000)	0.520 (0.263)	0.513 (0.287)	0.510 (0.290)
	Clayton	0.501 (0.000)	0.480 (0.239)	0.021 (0.038)	0.205 (0.201)
	Gumbel	0.479 (0.000)	0.460 (0.237)	0.549 (0.294)	0.294 (0.247)
	Frank	0.415 (0.000)	0.419 (0.232)	0.535 (0.311)	0.428 (0.287)
Clayton	Gaussian	0.003 (0.002)	0.008 (0.015)	0.312 (0.187)	0.248 (0.237)
	Clayton	0.520 (0.159)	0.535 (0.263)	0.519 (0.269)	0.501 (0.283)
	Gumbel	0.002 (0.002)	0.016 (0.024)	0.370 (0.222)	0.103 (0.139)
	Frank	0.008 (0.004)	0.040 (0.051)	0.424 (0.226)	0.265 (0.242)
Gumbel	Gaussian	0.082 (0.027)	0.095 (0.118)	0.109 (0.100)	0.390 (0.279)
	Clayton	0.035 (0.012)	0.214 (0.181)	0.000 (0.001)	0.101 (0.129)
	Gumbel	0.533 (0.110)	0.533 (0.270)	0.528 (0.264)	0.506 (0.287)
	Frank	0.113 (0.034)	0.340 (0.239)	0.417 (0.246)	0.463 (0.286)
Frank	Gaussian	0.242 (0.102)	0.129 (0.152)	0.104 (0.086)	0.380 (0.274)
	Clayton	0.536 (0.153)	0.400 (0.248)	0.000 (0.001)	0.173 (0.184)
	Gumbel	0.396 (0.135)	0.492 (0.265)	0.325 (0.227)	0.365 (0.267)
	Frank	0.509 (0.151)	0.508 (0.272)	0.506 (0.245)	0.486 (0.281)

Note: Applied to samples of size $n = 100$ for $d = 5$ dimensional copulae with dependence parameter $\tau = 0.5$.

Table 3. Estimated mean P-value (mean of $d!$ separate estimations based on the same data set) for approaches based on Rosenblatt's transformation. In parentheses the standard deviation over all permutations is given. All quoted values are averaged over 1000 simulations.

$\mathcal{H}_0$	$\mathcal{H}_1$	$\mathcal{A}_1^{(a)}$	$\mathcal{A}_1^{(b)}$	$\mathcal{A}_3$	$\mathcal{A}_8$
Gaussian	Gaussian	0.514 (0.000)	0.530 (0.057)	0.523 (0.000)	0.510 (0.284)
	Clayton	0.501 (0.000)	0.483 (0.056)	0.021 (0.000)	0.205 (0.194)
	Gumbel	0.479 (0.000)	0.458 (0.052)	0.559 (0.000)	0.294 (0.239)
	Frank	0.415 (0.000)	0.416 (0.048)	0.551 (0.000)	0.432 (0.282)
Clayton	Gaussian	0.002 (0.000)	0.008 (0.003)	0.318 (0.000)	0.250 (0.216)
	Clayton	0.517 (0.000)	0.535 (0.056)	0.524 (0.000)	0.501 (0.275)
	Gumbel	0.002 (0.000)	0.013 (0.003)	0.382 (0.000)	0.105 (0.125)
	Frank	0.008 (0.000)	0.038 (0.007)	0.436 (0.000)	0.262 (0.218)
Gumbel	Gaussian	0.080 (0.000)	0.089 (0.023)	0.104 (0.000)	0.390 (0.268)
	Clayton	0.036 (0.000)	0.205 (0.036)	0.000 (0.000)	0.100 (0.123)
	Gumbel	0.527 (0.000)	0.531 (0.061)	0.532 (0.000)	0.508 (0.281)
	Frank	0.112 (0.000)	0.342 (0.050)	0.421 (0.000)	0.461 (0.278)
Frank	Gaussian	0.240 (0.000)	0.129 (0.031)	0.109 (0.000)	0.381 (0.263)
	Clayton	0.541 (0.000)	0.395 (0.055)	0.000 (0.000)	0.170 (0.174)
	Gumbel	0.391 (0.000)	0.489 (0.059)	0.320 (0.000)	0.366 (0.257)
	Frank	0.502 (0.000)	0.510 (0.063)	0.501 (0.000)	0.485 (0.274)

Note: Applied to samples of size $n = 100$ for $d = 5$ dimensional copulae with dependence parameter $\tau = 0.5$.

performs very well for high dimensions and large sample sizes. To decide which approaches to consider, a preliminary test of ellipticity (see e.g. Huffera and Park (2007)) may also be helpful. The strong performance of approach $\mathcal{A}_9^{(b)}$ is very interesting and further research into the properties and power of this and other average approaches should be carried out.

When doing model evaluation, it is recommended to also examine various diagnostic tests such as g-o-f plots, e.g. plotting $S_4(w)$ with simulated null hypothesis confidence bands as done in Genest et al. (2006a). This may give valuable information on the fit of a copula. However, there is still an unsatisfied need for intuitive and informative diagnostic plots. Ideally such a plot should show, in some way and in case of rejection by the formal tests, which variable (i.e. which dimension) and/or which samples causes the rejection. Is it actually a deviation in the dependence structure between the variables or is the rejection due to some extreme samples? More research is needed on this topic.

Next, results were reported on the variation of the P-value estimates due to permutation order for approaches based on Rosenblatt's transformation. In general, one does not want a statistical testing procedure to give different values when running it several times on the same data set. However, for some of the approaches based on Rosenblatt's transformation, the estimated P-value will be different depending on which permuation order that is chosen for the variables. The practical consequence of this variation decreases as the P-value estimates approach critical/rejection levels. Hence, the author does not believe that the permutation effect is something to worry about. Also, as long as the permutation order is chosen in a random fashion, the results are not influenced in any particular direction.

The results concerning the permutation of variables also point in direction of important future research. The variation of P-value estimates also depends on the bootstrap parameters M and N_b. These parameters are usually, in a rather arbitrary way, set to what is believed to be large values. This is also the case in this paper. However, there has been no study of the effect that these choices may have on the power, and even more importantly the nominal levels of an approach. Originally, in the power studies of Section 4.1, a double bootstrap parameter $N_b = 2500$ was chosen for all combinations of dimension, sample size, dependence and alternative copula. However, for dimension $d = 8$ we observed some peculiar results, e.g. decreasing power as sample size increased. These peculiarities vanished when increasing N_b to 5000 for dimension $d = 8$. Choosing appropriately large values for these parameters and thus achieving proper nominal levels is crucial for any study and/or application of these g-o-f approaches. Hence, a study of the effects of these parameters and required minimum values would be highly valuable.

The computational aspect also deserves some attention. An important quality of approaches based on Rosenblatt's transform is computational efficiency. Approaches $\mathcal{A}_2$, $\mathcal{A}_4$ and $\mathcal{A}_5$ need computationally intensive double parametric bootstrap procedures to estimate P-values in some cases (e.g. for the elliptical copulae, in particular for higher dimensions and large sample sizes). Approaches based on Rosenblatt's transformation do not, in general, need this double bootstrap step, since after Rosenblatt's transformation, the null hypothesis is always the independence copula.

Finally, a word of warning. As emphasized in Genest et al. (2008), the asymptotics of several of the procedures presented here are not known. Hence, one cannot know for sure whether a bootstrap procedure will converge in every case. However, all the results so far on the performance of the proposed approaches and bootstrap procedures are comforting and strongly indicate the validity of the test procedures. Keep in mind though, the original approach and test procedure proposed by Breymann et al. (2003), which showed terrible performance in the study of Dobrić and Schmid (2007). This shows how wrong it can all go if our test procedure is not valid. Approaches $\mathcal{A}_2$ and $\mathcal{A}_4$, that turned out to be among the best in our study, both have known asymptotics and the bootstrap procedures for these approaches are well established from Quessy (2005), Genest et al. (2006a) and Genest and Rémillard (2008).

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A. Proof of permutation invariance of $\mathcal{A}_1^{(a)}$ under Gaussian copula null hypothesis

To prove that approach $\mathcal{A}_1^{(a)}$ is permutatoin invariant under the Gaussian copula null hypothesis, let us first look at how Rosenblatt's transformation is carried out. For the Gaussian copula null hypothesis, this transformation is easily computed using the Cholesky decomposition of the covariance matrix. Let $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ be a d-dimensional vector, where $\boldsymbol{\mu} = \mathbb{E}(\mathbf{X})$ and $\boldsymbol{\Sigma}$ is the $d \times d$ positive definite covariance matrix.

Since $\boldsymbol{\Sigma}$ is positive definite it can be written as $\boldsymbol{\Sigma} = \mathbf{A}^T\mathbf{A}$, where $\mathbf{A}$ is a lower triangular matrix and $\mathbf{A}^T$ denotes its transpose. Next, it is well known that $\mathbf{X}$ can be expressed as $\mathbf{X} = \boldsymbol{\mu} + \mathbf{A}^T\mathbf{Y}$ where $\mathbf{Y} \sim \mathcal{N}(0, \boldsymbol{I})$ and $\boldsymbol{I}$ is the d-dimensional identity matrix. I.e. $\mathbf{Y}$ is a vector of d i.i.d. standard normally distributed variables. Solving for $\mathbf{Y}$ gives $\mathbf{Y} = (\mathbf{A}^T)^{-1}(\mathbf{X} - \boldsymbol{\mu})$. We now see that the vector $\mathbf{V} = \Phi(\mathbf{Y})$ is i.i.d. $U(0,1)^d$ under the Gaussian null hypothesis.

For approach $\mathcal{A}_1^{(a)}$ we now need to compute $W_1 = \sum_{i=1}^d \Phi^{-1}(V_i)^2 = \sum_{i=1}^d Y_i^2 = \mathbf{Y}^T\mathbf{Y}$. We now proceed with the bivariate setting for simplicity but the proof can easily be extended to arbitrary dimension d. Consider the Cholesky decomposition of the covariance matrix $\boldsymbol{\Sigma} = \mathbf{A}^T\mathbf{A}$ in detail:

$$\boldsymbol{\Sigma}^1 = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & 0 \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^2 + a_{12}^2 & a_{12}a_{22} \\ a_{12}a_{22} & a_{22}^2 \end{pmatrix},$$

where the superscript 1 in $\boldsymbol{\Sigma}^1$ denotes permutation order 1. We see now that $a_{11} = \sqrt{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}/\sigma_2$, $a_{12} = \sigma_{12}/\sigma_2$ and $a_{22} = \sigma_2$. Next, we see that

$$(\mathbf{A}^T)^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & -\frac{a_{12}}{a_{11}a_{22}} \\ 0 & \frac{1}{a_{22}} \end{pmatrix}$$

and that

$$\mathbf{Y} = (\mathbf{A}^T)^{-1}(\mathbf{X} - \boldsymbol{\mu}) = \begin{pmatrix} \frac{1}{a_{11}}(X_1 - \mu_1) - \frac{a_{12}}{a_{11}a_{22}}(X_2 - \mu_2) \\ \frac{1}{a_{22}}(X_2 - \mu_2) \end{pmatrix}.$$

Now to compute $W_1^1 = \mathbf{Y}^T\mathbf{Y}$, superscript 1 denoting permutation order 1, we get

$$\begin{aligned} W_1^1 &= \frac{(X_1-\mu_1)^2}{a_{11}^2} + \frac{a_{12}^2}{a_{11}^2a_{22}^2}(X_2-\mu_2)^2 - \frac{2a_{12}}{a_{11}^2a_{22}}(X_1-\mu_1)(X_2-\mu_2) + \frac{(X_2-\mu_2)^2}{a_{22}^2} \\ &= \frac{(X_1-\mu_1)^2\sigma_2^2 + (X_2-\mu_2)^2\sigma_1^2 - 2(X_1-\mu_1)(X_2-\mu_2)\sigma_{12}}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \end{aligned}$$

by inserting σ's for the a's.

By doing the same exercise with permutation order 2 we first get

$$\boldsymbol{\Sigma}^2 = \begin{pmatrix} \sigma_2^2 & \sigma_{12} \\ \sigma_{12} & \sigma_1^2 \end{pmatrix}$$

and $a_{11} = \sqrt{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}/\sigma_1$, $a_{12} = \sigma_{12}/\sigma_1$ and $a_{22} = \sigma_1$. Next, in the same manner as above, it is easily shown that

$$W_1^2 = \frac{(X_2-\mu_2)^2\sigma_1^2 + (X_1-\mu_1)^2\sigma_2^2 - 2(X_1-\mu_1)(X_2-\mu_2)\sigma_{12}}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} = W_1^1.$$

Hence we have shown that approach $\mathcal{A}_1^{(a)}$ is permutation invariant under the Gaussian copula null hypothesis. This is not so for other weight functions or other null hypothesis copulae. The invariance stems from the use of Φ^{-1} which cancels out with the Φ in $\mathbf{V} = \Phi(\mathbf{Y})$ and the squaring $\Phi(V_i)^2$.

B. Derivation of a Cramér-von Mises statistic

Consider the Cramér–von Mises (CvM) statistic

$$T = n\int_0^1 \{\widehat{F}(w) - F(w)\}^2 \mathrm{d}F(w),$$

where $\widehat{F}(w) = \frac{1}{n+1}\sum_{j=1}^n I(X_j \leq t)$ is the empirical distribution function. Given a random sample $(x_1, \ldots, x_n)$, the empirical version $\widehat{T}$ of the CvM statistic can be derived as follows.

$$\begin{aligned}\widehat{T} =& n\int_0^1 \{\widehat{F}(w) - F(w)\}^2 \mathrm{d}F(w)\\ =& n\int_0^1 \widehat{F}(w)^2 \mathrm{d}F(w) - 2n\int_0^1 \widehat{F}(w)F(w)\mathrm{d}F(w) + n\int_0^1 F(w)^2\mathrm{d}F(w).\end{aligned}$$

Since $\widehat{F}(w)$ is constant and equal to $\widehat{F}(j/(n+1))$ between $j/(n+1)$ and $(j+1)/(n+1)$ for $j = 1, \ldots, n$, the first two integrals can be split into n smaller integrals:

$$\begin{aligned}\widehat{T} =& n\sum_{j=1}^n \int_{j/(n+1)}^{(j+1)/(n+1)} \widehat{F}\left(\frac{j}{n+1}\right)^2 \mathrm{d}F(w)\\ & -2n\sum_{j=1}^n \int_{j/(n+1)}^{(j+1)/(n+1)} \widehat{F}\left(\frac{j}{n+1}\right) F(w)\mathrm{d}F(w) + \frac{n}{3}\Big[F(w)^3\Big]_0^1\\ =& \frac{n}{3} + n\sum_{j=1}^n \widehat{F}\left(\frac{j}{n+1}\right)^2 \left\{F\left(\frac{j+1}{n+1}\right) - F\left(\frac{j}{n+1}\right)\right\}\\ & -n\sum_{j=1}^n \widehat{F}\left(\frac{j}{n+1}\right)\left\{F\left(\frac{j+1}{n+1}\right)^2 - F\left(\frac{j}{n+1}\right)^2\right\}.\end{aligned}$$

For approach $\mathcal{A}_1$ the test observator $S_1(w)$ is $U[0,1]$ under the null hypothesis. Hence $F(w) = w$ and we easily see that $\widehat{T}$ reduces to

$$\widehat{T}' = \frac{n}{3} + \frac{n}{n+1}\sum_{j=1}^n \widehat{F}\left(\frac{j}{n+1}\right)^2 - \frac{n}{(n+1)^2}\sum_{j=1}^n (2j+1)\widehat{F}\left(\frac{j}{n+1}\right).$$

C. Power results from numerical experiments

Table C.4. Percentage of rejections (at 5% significance level) of the Gaussian copula.

d	n	τ	True copula	$\mathcal{A}_1^{(a)}$	$\mathcal{A}_1^{(b)}$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$	$\mathcal{A}_8$	$\mathcal{A}_9^{(a)}$	$\mathcal{A}_9^{(b)}$
2	100	0.2	Gaussian	*5.3*	*5.0*	*5.0*	*4.6*	*5.4*	*5.7*	–	*4.7*	*5.2*	*5.0*	*5.1*
			Student ($\nu = 6$)	0.9	4.2	7.0	**8.8**	6.1	5.3	–	5.6	6.0	3.3	6.4
			Clayton	2.6	5.0	19.7	19.6	19.9	15.6	–	7.1	6.9	10.6	**24.0**
			Gumbel	1.9	4.6	10.7	3.6	**11.6**	8.4	–	6.2	5.9	4.9	9.7
			Frank	3.4	3.2	6.0	**7.4**	6.0	6.2	–	5.4	5.5	3.4	6.1
		0.4	Gaussian	*5.2*	*5.0*	*4.7*	*5.4*	*4.8*	*4.7*	–	*5.0*	*4.7*	*5.0*	*4.9*
			Student ($\nu = 6$)	1.3	2.4	5.9	**11.6**	4.8	3.9	–	5.3	5.8	2.3	6.4
			Clayton	1.1	2.5	57.4	59.6	49.7	33.7	–	14.9	15.8	22.2	**63.9**
			Gumbel	1.3	2.6	**19.1**	5.0	18.5	8.2	–	7.0	7.9	4.1	16.2
			Frank	0.8	1.2	10.6	11.6	10.1	8.9	–	6.1	6.3	1.5	**11.8**
	500	0.2	Gaussian	*4.7*	*4.9*	*5.2*	*4.8*	*5.2*	*5.1*	–	*5.1*	*4.9*	*4.9*	*5.0*
			Student ($\nu = 6$)	19.5	16.9	10.0	16.9	8.4	8.5	–	10.3	9.8	**21.4**	10.0
			Clayton	2.0	5.8	72.5	71.3	71.9	57.2	–	23.8	20.3	56.5	**79.5**
			Gumbel	2.5	6.9	33.2	8.5	33.9	25.8	–	12.3	11.1	21.2	**34.3**
			Frank	2.2	2.9	11.4	**21.9**	11.1	9.9	–	7.6	8.1	5.8	14.5
		0.4	Gaussian	*5.0*	*5.0*	*4.6*	*5.4*	*4.9*	*4.8*	–	*4.9*	*5.5*	*5.1*	*4.8*
			Student ($\nu = 6$)	23.8	12.5	8.2	**30.5**	6.6	6.9	–	10.1	12.6	20.6	12.0
			Clayton	6.8	4.3	99.8	**100**	99.6	96.2	–	78.1	84.3	99.0	99.9
			Gumbel	8.8	6.0	**65.3**	18.9	62.9	39.8	–	26.4	32.4	42.3	**65.3**
			Frank	15.1	12.2	36.9	**60.7**	33.4	26.4	–	17.0	20.6	36.9	52.1
4	100	0.2	Gaussian	*4.8*	*5.0*	*4.6*	*4.8*	*4.8*	*5.3*	–	*5.6*	*5.0*	*5.0*	*4.9*
			Student ($\nu = 6$)	5.1	6.5	8.9	**15.4**	8.5	7.0	–	6.7	6.6	7.5	9.7
			Clayton	1.1	5.0	45.6	30.5	52.5	19.2	–	9.4	7.0	20.2	**55.9**
			Gumbel	1.2	3.1	12.8	0.7	42.5	**56.4**	–	13.9	8.8	13.2	34.9
			Frank	2.0	1.4	1.8	3.0	12.2	**19.6**	–	7.5	6.8	2.0	8.4
		0.4	Gaussian	*4.5*	*4.8*	*5.2*	*5.4*	*5.1*	*5.1*	–	*4.9*	*5.3*	*4.9*	*5.3*
			Student ($\nu = 6$)	9.2	3.7	8.6	**24.4**	6.1	5.3	–	6.9	7.1	7.5	8.1
			Clayton	1.1	1.8	**90.8**	80.4	84.0	45.6	–	27.9	18.3	48.8	90.1
			Gumbel	1.5	1.7	41.0	3.6	**52.0**	48.7	–	25.8	15.4	17.1	50.1
			Frank	1.6	2.2	10.1	7.3	**23.6**	20.6	–	12.6	8.3	5.6	21.2
	500	0.2	Gaussian	*5.8*	*5.3*	*5.3*	*5.0*	*4.8*	*4.9*	–	*5.0*	*5.5*	*4.9*	*4.7*
			Student ($\nu = 6$)	**98.5**	71.8	16.5	47.1	11.2	12.6	–	13.6	15.0	96.5	15.7
			Clayton	4.3	7.7	99.0	94.4	98.0	88.4	–	39.3	22.2	94.6	**99.2**
			Gumbel	8.0	5.9	84.2	48.0	97.7	**98.5**	–	70.3	34.7	92.3	98.0
			Frank	3.6	6.6	25.4	5.0	64.3	**66.2**	–	20.3	17.2	39.1	63.8
		0.4	Gaussian	*4.7*	*4.7*	*4.8*	*4.9*	*4.7*	*4.8*	–	*5.1*	*5.0*	*4.4*	*4.6*
			Student ($\nu = 6$)	**98.1**	67.5	11.6	72.1	8.0	8.8	–	16.4	18.7	94.0	13.8
			Clayton	44.3	13.2	**100**	**100**	**100**	99.9	–	97.2	91.2	**100**	**100**
			Gumbel	63.2	34.7	98.9	70.1	99.6	98.1	–	95.5	77.4	99.4	**99.8**
			Frank	79.3	74.2	73.2	19.5	88.6	74.5	–	61.2	40.7	**97.4**	90.6
8	100	0.2	Gaussian	*5.0*	*5.2*	*5.9*	*4.7*	*5.8*	*5.2*	–	*5.3*	*5.2*	*5.4*	*5.7*
			Student ($\nu = 6$)	**40.4**	16.4	9.8	15.0	12.3	7.7	–	7.9	6.9	35.9	12.4
			Clayton	0.7	4.1	48.7	24.3	**66.0**	1.2	–	11.8	6.6	19.5	65.5
			Gumbel	0.6	1.7	22.0	2.3	61.5	**98.3**	–	56.9	13.8	14.0	56.1
			Frank	0.4	0.6	3.8	1.3	7.3	**56.0**	–	14.4	7.2	0.6	4.7
		0.4	Gaussian	*5.1*	*5.2*	*5.0*	*4.6*	*5.3*	*5.7*	–	*5.5*	*5.1*	*5.3*	*5.1*
			Student ($\nu = 6$)	**51.7**	16.1	8.3	17.6	7.4	6.1	–	8.0	8.5	39.2	7.8
			Clayton	1.6	2.4	**96.6**	49.2	93.3	28.1	–	40.4	19.9	59.9	95.0
			Gumbel	16.2	10.1	70.5	2.7	78.4	**92.8**	–	67.9	28.1	52.7	78.6
			Frank	4.8	8.3	19.6	2.9	**28.7**	23.9	–	26.7	7.5	14.6	25.7
	500	0.2	Gaussian	*5.5*	*4.8*	*4.4*	*5.1*	*4.8*	*5.4*	–	*5.2*	*5.1*	*4.6*	*4.8*
			Student ($\nu = 6$)	**100**	99.9	23.7	56.4	19.1	11.8	–	21.7	20.9	**100**	21.3
			Clayton	11.8	12.9	**100**	74.3	99.7	84.8	–	50.5	13.6	97.2	99.9
			Gumbel	30.0	13.4	**100**	71.7	**100**	**100**	–	**100**	63.0	99.9	**100**
			Frank	22.9	38.3	99.8	10.5	98.4	**99.9**	–	69.6	19.4	90.7	99.8
		0.4	Gaussian	*4.9*	*5.4*	*4.9*	*5.2*	*5.4*	*5.1*	–	*4.7*	*5.9*	*5.1*	*5.2*
			Student ($\nu = 6$)	**100**	99.8	16.9	71.5	12.2	10.6	–	21.4	32.0	**100**	13.7
			Clayton	78.0	52.6	**100**	99.8	**100**	**100**	–	99.2	81.5	**100**	**100**
			Gumbel	**100**	98.7	**100**	33.9	**100**	**100**	–	**100**	94.7	**100**	**100**
			Frank	99.5	99.5	**100**	1.9	99.8	95.6	–	97.3	37.7	**100**	**100**

Note: Numbers in *italics* are nominal levels and should correspond to the size of 5%. Numbers in **bold** indicate the best performing approach.

Table C.5. Percentage of rejections (at 5% significance level) of the Student copula.

d	n	τ	True copula	$\mathcal{A}_1^{(a)}$	$\mathcal{A}_1^{(b)}$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$	$\mathcal{A}_8$	$\mathcal{A}_9^{(a)}$	$\mathcal{A}_9^{(b)}$
2	100	0.2	Gaussian	**5.7**	5.4	4.9	4.0	5.0	5.2	–	5.6	5.3	5.6	4.8
			Student ($\nu = 6$)	*4.4*	*4.6*	*4.8*	*4.1*	*5.1*	*4.8*	–	*5.1*	*5.0*	*4.6*	*4.8*
			Clayton	4.8	5.3	19.2	11.0	20.1	17.2	–	7.3	6.8	15.4	**21.3**
			Gumbel	4.7	5.1	9.2	4.9	**10.5**	7.0	–	5.9	5.8	7.6	10.1
			Frank	4.9	5.4	6.0	4.4	6.6	**7.1**	–	5.8	5.7	6.5	6.6
		0.4	Gaussian	4.7	5.4	4.9	4.0	5.2	5.4	–	**5.7**	4.9	5.2	4.8
			Student ($\nu = 6$)	*4.1*	*4.5*	*4.2*	*4.4*	*4.8*	*5.1*	–	*4.9*	*4.9*	*4.4*	*4.4*
			Clayton	4.2	4.9	55.0	31.7	53.3	41.1	–	15.4	14.8	39.9	**57.3**
			Gumbel	4.4	5.0	17.2	6.1	**18.7**	9.1	–	7.2	7.4	10.5	17.5
			Frank	2.9	3.4	11.8	5.3	**12.5**	10.5	–	7.5	6.3	6.9	11.6
	500	0.2	Gaussian	5.8	5.8	5.1	5.1	5.0	5.6	–	5.8	5.5	**6.0**	5.3
			Student ($\nu = 6$)	*5.1*	*5.1*	*4.5*	*4.5*	*4.5*	*5.3*	–	*5.1*	*5.2*	*4.8*	*4.6*
			Clayton	5.6	4.8	69.9	60.4	72.4	61.3	–	22.0	19.9	65.7	**77.5**
			Gumbel	5.2	5.3	28.6	18.6	30.0	19.7	–	11.0	10.0	23.5	**33.2**
			Frank	5.2	6.3	12.3	8.3	12.7	12.6	–	7.4	7.8	11.6	**13.4**
		0.4	Gaussian	**5.6**	5.2	4.5	5.3	5.0	5.5	–	5.2	4.9	5.4	5.0
			Student ($\nu = 6$)	*4.9*	*4.6*	*5.3*	*4.4*	*4.5*	*4.8*	–	*4.7*	*5.0*	*4.7*	*4.6*
			Clayton	6.4	7.0	99.8	99.6	99.6	97.7	–	74.6	78.4	99.5	**99.9**
			Gumbel	4.5	5.1	61.7	40.0	61.2	34.1	–	22.4	24.1	49.2	**68.3**
			Frank	11.6	5.9	41.2	15.4	40.4	31.7	–	17.2	14.2	36.0	**44.8**

Note: Numbers in *italics* are nominal levels and should correspond to the size of 5%. Numbers in **bold** indicate the best performing approach.

Table C.6. Percentage of rejections (at 5% significance level) of the Clayton copula.

d	n	τ	True copula	$\mathcal{A}_1^{(a)}$	$\mathcal{A}_1^{(b)}$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$	$\mathcal{A}_8$	$\mathcal{A}_9^{(a)}$	$\mathcal{A}_9^{(b)}$
2	100	0.2	Gaussian	7.5	7.3	21.3	6.6	**23.2**	14.5	20.9	7.3	6.9	20.8	22.4
			Student ($\nu = 6$)	8.0	8.5	23.8	8.4	**24.1**	16.3	15.9	7.5	7.0	21.0	23.7
			Clayton	*4.9*	*5.1*	*5.0*	*5.2*	*5.0*	*5.2*	*4.5*	*5.2*	*5.2*	*5.0*	*5.1*
			Gumbel	6.2	9.4	46.7	13.0	**47.3**	32.3	40.4	12.4	11.1	41.2	47.1
			Frank	7.0	6.9	24.6	6.4	27.1	16.3	**30.3**	8.6	7.4	25.1	25.8
		0.4	Gaussian	24.0	26.7	58.9	26.4	58.2	33.7	62.1	16.6	15.3	**66.5**	60.6
			Student ($\nu = 6$)	13.4	19.0	**60.6**	16.0	58.4	35.1	53.6	15.4	13.7	58.2	57.3
			Clayton	*4.4*	*4.8*	*4.8*	*5.4*	*4.9*	*4.9*	*4.8*	*4.7*	*4.8*	*4.6*	*4.8*
			Gumbel	29.7	38.9	91.6	41.2	90.6	70.1	90.2	34.9	31.7	**92.0**	90.2
			Frank	24.1	19.2	64.8	24.2	66.2	35.6	**84.3**	19.3	16.5	77.0	65.6
	500	0.2	Gaussian	20.6	13.3	78.7	44.8	70.2	52.9	**85.9**	24.0	20.5	68.5	75.3
			Student ($\nu = 6$)	26.9	23.3	**82.1**	33.4	73.7	64.8	68.5	26.1	22.2	76.1	77.6
			Clayton	*5.2*	*5.1*	*5.0*	*4.8*	*5.1*	*5.4*	*5.1*	*5.3*	*4.5*	*4.8*	*5.2*
			Gumbel	12.6	23.2	**99.2**	84.9	97.9	94.0	99.0	60.1	52.0	97.2	98.6
			Frank	18.8	9.0	86.6	42.9	82.2	63.4	**97.6**	30.4	22.7	78.3	84.8
		0.4	Gaussian	94.8	85.6	**100**	99.5	99.7	95.5	**100**	77.7	82.3	99.9	99.9
			Student ($\nu = 6$)	65.3	71.4	**99.9**	89.7	99.6	97.3	99.8	74.7	74.9	99.8	99.8
			Clayton	*5.3*	*5.1*	*5.0*	*5.2*	*4.7*	*4.8*	*4.9*	*4.7*	*4.4*	*5.0*	*4.7*
			Gumbel	98.4	97.8	**100**	**100**	**100**	**100**	**100**	99.4	99.5	**100**	**100**
			Frank	97.8	69.9	**100**	99.4	99.9	96.7	**100**	84.6	86.8	**100**	**100**
4	100	0.2	Gaussian	10.8	10.6	37.4	3.2	38.5	39.1	**49.8**	10.6	6.5	49.2	37.9
			Student ($\nu = 6$)	27.1	21.3	48.4	17.8	37.7	42.2	37.7	10.1	7.3	**57.2**	42.5
			Clayton	*4.7*	*5.1*	*5.3*	*5.6*	*5.2*	*5.1*	*4.6*	*6.3*	*4.7*	*5.0*	*5.2*
			Gumbel	8.8	12.0	64.4	3.0	91.1	**94.1**	81.5	31.9	14.0	88.4	88.6
			Frank	7.7	6.5	36.0	1.4	**74.7**	68.9	73.0	15.1	7.2	72.8	68.8
		0.4	Gaussian	78.3	65.7	89.8	3.0	83.0	73.9	91.6	31.0	16.7	**95.2**	84.3
			Student ($\nu = 6$)	53.9	45.7	**92.9**	6.1	82.6	76.0	86.2	29.9	15.8	92.2	85.6
			Clayton	*5.2*	*4.7*	*5.6*	*5.5*	*5.2*	*5.1*	*4.5*	*5.3*	*4.9*	*5.1*	*5.3*
			Gumbel	79.1	62.1	99.3	4.9	99.8	99.8	99.8	80.8	40.1	**99.9**	99.8
			Frank	68.7	37.9	91.4	3.2	97.0	84.8	**99.6**	52.4	15.1	99.3	96.3
	500	0.2	Gaussian	89.6	38.1	99.4	18.1	97.0	91.2	**99.9**	38.8	23.0	99.4	98.0
			Student ($\nu = 6$)	93.7	76.9	99.9	89.7	95.8	94.5	97.9	44.1	30.8	**100**	98.7
			Clayton	*4.8*	*4.7*	*5.2*	*5.6*	*5.6*	*4.7*	*5.0*	*4.8*	*5.3*	*5.1*	*5.6*
			Gumbel	71.1	37.8	**100**	80.3	**100**	**100**	**100**	97.8	83.4	**100**	**100**
			Frank	82.6	11.8	99.8	14.5	**100**	99.9	**100**	67.9	24.8	**100**	**100**
		0.4	Gaussian	**100**	**100**	**100**	99.7	**100**	99.9	**100**	97.4	95.5	**100**	**100**
			Student ($\nu = 6$)	**100**	99.8	**100**	80.0	**100**	**100**	**100**	96.9	90.1	**100**	**100**
			Clayton	*4.9*	*5.2*	*5.3*	*5.7*	*5.6*	*5.2*	*5.6*	*4.8*	*5.5*	*5.1*	*5.4*
			Gumbel	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
			Frank	**100**	99.0	**100**	99.9	**100**	**100**	**100**	**100**	93.6	**100**	**100**
8	100	0.2	Gaussian	14.3	12.6	29.9	9.9	21.4	53.5	**82.6**	8.1	6.6	74.2	22.3
			Student ($\nu = 6$)	57.8	61.0	44.3	40.9	20.2	54.3	65.9	9.3	8.6	**85.5**	24.4
			Clayton	*5.5*	*5.0*	*5.2*	*5.5*	*5.6*	*5.4*	*4.3*	*4.7*	*5.2*	*5.1*	*5.5*
			Gumbel	7.6	10.5	63.2	52.6	91.9	**100**	98.0	68.7	26.5	97.0	90.8
			Frank	3.2	6.0	16.6	4.2	74.8	96.5	**96.7**	20.4	6.3	93.4	68.9
		0.4	Gaussian	97.5	91.7	96.9	2.5	87.1	89.0	98.2	34.8	10.9	**99.1**	90.2
			Student ($\nu = 6$)	86.3	80.5	**98.4**	29.5	86.1	89.4	96.0	32.4	10.7	97.7	91.4
			Clayton	*5.7*	*5.4*	*4.8*	*5.1*	*4.7*	*4.8*	*4.6*	*5.3*	*5.0*	*4.7*	*4.7*
			Gumbel	93.0	82.2	99.8	19.9	**100**	**100**	**100**	97.3	43.4	**100**	**100**
			Frank	85.2	62.8	93.7	0.6	99.6	97.7	**100**	76.5	8.1	**100**	99.6
	500	0.2	Gaussian	**100**	71.6	**100**	24.9	98.9	97.4	**100**	41.8	17.0	**100**	99.5
			Student ($\nu = 6$)	**100**	**100**	**100**	99.3	96.7	98.1	**100**	50.8	32.0	**100**	99.3
			Clayton	*5.3*	*4.8*	*5.0*	*4.8*	*4.9*	*5.3*	*4.6*	*5.3*	*5.4*	*5.4*	*4.7*
			Gumbel	98.3	40.7	**100**	96.6	**100**	**100**	**100**	**100**	96.8	**100**	**100**
			Frank	99.9	11.0	**100**	3.7	**100**	**100**	**100**	92.8	15.5	**100**	**100**
		0.4	Gaussian	**100**	**100**	**100**	96.1	**100**	**100**	**100**	98.7	84.4	**100**	**100**
			Student ($\nu = 6$)	**100**	**100**	**100**	93.2	**100**	**100**	**100**	98.7	78.1	**100**	**100**
			Clayton	*4.5*	*4.8*	*4.8*	*4.9*	*4.9*	*5.2*	*5.1*	*5.5*	*4.9*	*4.8*	*4.8*
			Gumbel	**100**	**100**	**100**	88.5	**100**	**100**	**100**	**100**	**100**	**100**	**100**
			Frank	**100**	**100**	**100**	69.5	**100**	**100**	**100**	**100**	76.0	**100**	**100**

Note: Numbers in *italics* are nominal levels and should correspond to the size of 5%. Numbers in **bold** indicate the best performing approach.

Table C.7. Percentage of rejections (at 5% significance level) of the Gumbel copula.

d	n	τ	True copula	$\mathcal{A}_1^{(a)}$	$\mathcal{A}_1^{(b)}$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$	$\mathcal{A}_8$	$\mathcal{A}_9^{(a)}$	$\mathcal{A}_9^{(b)}$
2	100	0.2	Gaussian	7.7	6.6	9.9	7.3	9.6	9.6	–	6.4	6.6	**10.2**	9.8
			Student ($\nu = 6$)	7.1	6.2	**11.2**	9.8	9.0	7.6	–	5.9	6.2	8.8	10.4
			Clayton	5.9	6.5	45.8	31.1	44.0	35.1	–	12.3	10.8	33.1	**47.5**
			Gumbel	*5.3*	*5.1*	*5.1*	*4.9*	*5.1*	*5.1*	–	*5.1*	*5.3*	*5.1*	*4.9*
			Frank	6.7	5.2	12.1	8.0	11.3	**13.3**	–	7.4	6.8	10.4	11.7
		0.4	Gaussian	11.4	11.2	17.5	8.9	16.4	13.7	–	8.1	7.2	**19.1**	17.6
			Student ($\nu = 6$)	5.8	6.2	**20.2**	15.2	16.1	11.3	–	7.5	6.7	13.9	19.7
			Clayton	8.1	14.0	**92.6**	75.4	89.8	75.3	–	34.7	31.4	83.4	**92.6**
			Gumbel	*4.8*	*4.6*	*4.8*	*5.1*	*4.9*	*4.7*	–	*4.7*	*5.2*	*4.8*	*5.0*
			Frank	8.1	7.1	**28.7**	9.4	24.8	24.3	–	10.3	9.0	20.9	25.7
	500	0.2	Gaussian	19.9	9.8	**37.0**	23.9	29.2	26.9	–	11.7	10.2	31.4	33.1
			Student ($\nu = 6$)	16.6	11.6	**39.1**	33.7	25.2	17.3	–	11.8	10.2	27.7	30.8
			Clayton	8.4	10.3	**99.6**	98.5	98.5	95.9	–	57.5	51.5	97.1	99.3
			Gumbel	*4.7*	*4.6*	*5.1*	*4.8*	*4.6*	*5.1*	–	*5.0*	*4.6*	*4.6*	*4.6*
			Frank	16.0	7.4	**53.9**	30.7	38.5	42.6	–	16.2	12.7	37.1	44.3
		0.4	Gaussian	49.9	32.4	**74.1**	38.4	61.6	46.8	–	25.4	28.9	73.8	67.7
			Student ($\nu = 6$)	9.0	10.8	**74.1**	56.7	57.3	36.0	–	20.9	21.1	53.0	68.4
			Clayton	43.6	57.8	**100**	**100**	**100**	**100**	–	99.3	99.6	**100**	**100**
			Gumbel	*5.4*	*4.9*	*5.2*	*5.5*	*5.0*	*5.0*	–	*4.8*	*5.2*	*5.0*	*4.9*
			Frank	45.3	13.8	**95.5**	47.8	85.1	82.2	–	44.4	42.1	86.2	89.2
4	100	0.2	Gaussian	6.8	13.0	54.7	43.4	51.1	24.0	–	14.9	7.5	41.6	**57.3**
			Student ($\nu = 6$)	24.9	24.8	56.8	55.7	52.8	21.1	–	13.0	8.8	58.7	**60.1**
			Clayton	3.4	15.1	89.6	85.4	97.1	82.2	–	29.9	10.1	90.6	**97.2**
			Gumbel	*5.0*	*4.9*	*5.0*	*4.5*	*5.0*	*5.3*	–	*5.0*	*5.6*	*4.8*	*5.0*
			Frank	4.6	5.4	22.2	13.1	29.2	**30.6**	–	12.6	5.5	18.6	30.0
		0.4	Gaussian	29.7	36.6	66.7	44.0	59.9	33.7	–	28.8	9.2	**70.5**	65.0
			Student ($\nu = 6$)	15.1	22.0	68.0	66.1	60.7	30.2	–	26.2	9.9	60.0	**68.9**
			Clayton	26.8	29.9	99.9	99.1	**100**	98.8	–	82.4	32.8	99.8	**100**
			Gumbel	*5.0*	*5.0*	*5.0*	*5.2*	*5.1*	*5.1*	–	*5.0*	*5.4*	*5.5*	*5.0*
			Frank	17.8	9.0	51.4	12.5	54.3	**56.1**	–	26.2	7.3	46.5	53.7
	500	0.2	Gaussian	75.9	59.1	**99.4**	98.5	98.3	96.0	–	68.4	19.5	**99.4**	99.2
			Student ($\nu = 6$)	92.0	88.5	99.1	99.7	97.7	94.5	–	67.4	27.3	**100**	99.2
			Clayton	34.2	64.9	**100**	**100**	**100**	**100**	–	98.1	53.3	**100**	**100**
			Gumbel	*4.7*	*4.8*	*4.8*	*4.6*	*4.7*	*5.0*	–	*4.7*	*4.2*	*4.6*	*4.7*
			Frank	47.7	10.0	86.6	47.5	92.7	**98.1**	–	58.0	9.8	93.2	94.0
		0.4	Gaussian	99.9	98.2	**100**	99.7	99.6	97.6	–	95.9	54.8	**100**	99.9
			Student ($\nu = 6$)	86.1	91.3	**100**	**100**	99.6	97.1	–	93.9	60.2	**100**	**100**
			Clayton	**100**	95.7	**100**	**100**	**100**	**100**	–	**100**	99.8	**100**	**100**
			Gumbel	*4.7*	*5.1*	*4.9*	*5.3*	*5.1*	*4.8*	–	*4.6*	*5.1*	*4.8*	*5.2*
			Frank	99.4	31.8	99.9	58.9	99.8	**100**	–	93.0	23.7	**100**	99.9
8	100	0.2	Gaussian	1.0	30.0	89.8	73.2	87.1	29.9	–	37.6	6.7	50.0	**90.4**
			Student ($\nu = 6$)	52.3	70.3	89.4	76.6	86.2	30.9	–	36.1	8.3	**91.9**	89.9
			Clayton	0.2	29.9	93.6	95.4	**99.8**	81.2	–	53.3	8.6	89.3	99.7
			Gumbel	*5.4*	*5.1*	*4.1*	*4.8*	*4.9*	*4.8*	–	*4.6*	*5.1*	*5.1*	*4.8*
			Frank	0.3	4.3	14.6	10.3	**40.4**	19.4	–	28.4	5.5	3.6	36.8
		0.4	Gaussian	36.8	68.2	**98.1**	72.3	90.2	50.3	–	70.1	6.8	93.7	93.7
			Student ($\nu = 6$)	45.3	65.7	**97.8**	83.8	90.8	51.8	–	65.0	11.7	94.1	94.6
			Clayton	38.5	45.9	**100**	99.6	**100**	99.9	–	98.2	42.0	**100**	**100**
			Gumbel	*5.2*	*5.1*	*5.3*	*5.1*	*5.3*	*5.4*	–	*5.0*	*5.5*	*5.2*	*5.4*
			Frank	16.0	8.7	54.3	9.6	**67.1**	63.5	–	53.4	4.9	42.5	66.2
	500	0.2	Gaussian	99.9	99.1	**100**	**100**	**100**	**100**	–	99.2	14.8	**100**	**100**
			Student ($\nu = 6$)	**100**	**100**	**100**	**100**	**100**	**100**	–	98.9	31.7	**100**	**100**
			Clayton	79.4	98.9	**100**	**100**	**100**	**100**	–	**100**	33.0	**100**	**100**
			Gumbel	*5.1*	*4.9*	*4.1*	*4.8*	*5.1*	*5.2*	–	*4.3*	*4.8*	*5.2*	*5.0*
			Frank	78.6	18.6	90.1	36.7	99.9	**100**	–	93.7	7.0	99.2	99.9
		0.4	Gaussian	**100**	**100**	**100**	**100**	**100**	**100**	–	**100**	37.5	**100**	**100**
			Student ($\nu = 6$)	**100**	**100**	**100**	**100**	**100**	**100**	–	**100**	67.5	**100**	**100**
			Clayton	**100**	99.9	**100**	**100**	**100**	**100**	–	**100**	99.7	**100**	**100**
			Gumbel	*5.3*	*4.9*	*5.1*	*5.3*	*5.2*	*5.4*	–	*4.9*	*5.0*	*5.2*	*5.1*
			Frank	**100**	48.8	**100**	35.6	**100**	**100**	–	99.8	9.5	**100**	**100**

Note: Numbers in *italics* are nominal levels and should correspond to the size of 5%. Numbers in **bold** indicate the best performing approach.

Table C.8. Percentage of rejections (at 5% significance level) of the Frank copula.

d	n	τ	True copula	$\mathcal{A}_1^{(a)}$	$\mathcal{A}_1^{(b)}$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$	$\mathcal{A}_8$	$\mathcal{A}_9^{(a)}$	$\mathcal{A}_9^{(b)}$
2	100	0.2	Gaussian	5.8	5.5	6.0	**7.5**	6.9	6.6	–	4.9	5.1	6.2	7.4
			Student ($\nu = 6$)	10.6	8.4	8.8	9.9	8.9	7.9	–	6.0	5.7	**11.9**	10.1
			Clayton	5.1	5.3	24.4	21.3	26.2	18.5	–	7.9	7.4	17.4	**29.4**
			Gumbel	5.2	6.0	13.5	8.8	14.2	11.4	–	6.3	6.3	10.0	**14.9**
			Frank	*5.8*	*5.6*	*5.5*	*7.3*	*5.6*	*5.4*	–	*5.4*	*4.8*	*5.7*	*5.9*
		0.4	Gaussian	12.2	9.1	9.4	9.2	9.5	6.8	–	5.6	6.5	**13.1**	10.7
			Student ($\nu = 6$)	8.2	6.4	13.7	10.4	13.3	9.4	–	6.2	7.1	12.0	**14.7**
			Clayton	6.8	5.2	65.4	47.5	62.4	34.6	–	15.9	16.9	46.6	**68.2**
			Gumbel	6.5	6.0	**29.1**	9.6	26.0	15.7	–	8.4	9.1	18.0	26.6
			Frank	*5.9*	*4.8*	*4.9*	*6.3*	*5.2*	*4.7*	–	*4.1*	*5.1*	*5.3*	*5.3*
	500	0.2	Gaussian	7.6	6.7	11.2	**15.3**	10.3	10.3	–	6.7	7.3	10.3	11.8
			Student ($\nu = 6$)	47.8	26.9	28.0	20.5	26.5	25.2	–	12.4	13.4	**48.0**	29.2
			Clayton	7.6	7.1	87.7	81.0	84.2	66.4	–	27.5	27.5	74.3	**87.8**
			Gumbel	11.4	10.3	**55.6**	31.9	44.5	41.8	–	15.1	15.9	41.1	49.2
			Frank	*5.5*	*4.9*	*4.5*	*7.2*	*5.4*	*5.1*	–	*4.6*	*5.4*	*4.9*	*5.5*
		0.4	Gaussian	30.3	23.1	42.5	35.1	32.7	23.2	–	14.0	14.9	**47.5**	42.2
			Student ($\nu = 6$)	20.9	14.5	**68.5**	28.6	57.1	46.2	–	22.3	21.5	58.9	63.8
			Clayton	11.9	9.5	**100**	99.9	**100**	97.6	–	83.9	85.2	99.9	**100**
			Gumbel	9.9	12.2	**95.2**	47.5	85.8	77.3	–	41.7	41.2	81.2	89.9
			Frank	*6.0*	*4.8*	*4.2*	*6.4*	*4.7*	*4.0*	–	*4.6*	*5.0*	*4.9*	*5.0*
4	100	0.2	Gaussian	4.8	9.3	27.6	27.0	24.8	10.3	–	6.9	6.9	18.2	**29.8**
			Student ($\nu = 6$)	44.0	25.9	40.0	41.1	36.8	20.3	–	8.2	7.7	**59.2**	44.5
			Clayton	6.5	8.5	68.0	75.0	87.1	41.9	–	13.2	8.5	71.9	**88.4**
			Gumbel	10.2	5.3	19.6	3.9	33.8	**50.5**	–	11.2	7.2	27.3	31.1
			Frank	*5.5*	*5.3*	*4.5*	*4.9*	*4.8*	*4.7*	–	*5.2*	*5.1*	*5.2*	*4.8*
		0.4	Gaussian	14.1	29.4	30.1	33.1	31.3	18.4	–	10.8	7.6	**43.9**	37.3
			Student ($\nu = 6$)	18.5	16.7	47.4	53.0	43.3	29.2	–	13.0	9.3	49.8	**53.6**
			Clayton	4.5	9.8	95.5	97.5	98.0	62.1	–	47.1	19.4	93.8	**98.8**
			Gumbel	9.7	5.1	58.0	7.2	54.7	**65.3**	–	21.3	9.1	44.0	56.6
			Frank	*5.6*	*4.8*	*5.4*	*5.4*	*5.3*	*5.7*	–	*5.2*	*4.6*	*5.4*	*5.5*
	500	0.2	Gaussian	13.4	38.1	**86.1**	79.1	66.0	57.7	–	19.8	15.9	77.3	76.2
			Student ($\nu = 6$)	99.0	90.2	97.4	95.7	88.3	88.7	–	34.3	27.9	**99.9**	95.2
			Clayton	11.2	31.1	**100**	**100**	**100**	99.7	–	66.7	37.3	**100**	**100**
			Gumbel	26.6	7.8	84.7	22.0	91.9	**97.5**	–	56.8	25.5	91.2	92.5
			Frank	*5.6*	*5.4*	*5.1*	*4.9*	*4.4*	*5.6*	–	*4.9*	*5.0*	*5.8*	*4.5*
		0.4	Gaussian	78.9	93.7	98.3	95.3	90.9	74.2	–	58.9	40.3	**99.9**	95.7
			Student ($\nu = 6$)	72.0	78.8	99.9	99.6	98.6	95.8	–	72.2	52.2	**100**	99.6
			Clayton	8.0	36.9	**100**	**100**	**100**	**100**	–	99.9	96.5	**100**	**100**
			Gumbel	35.0	6.9	**99.9**	51.9	99.7	**99.9**	–	91.5	54.4	99.7	99.8
			Frank	*4.9*	*5.1*	*5.3*	*6.0*	*5.0*	*5.1*	–	*5.7*	*4.8*	*5.0*	*5.3*
8	100	0.2	Gaussian	1.0	20.5	**81.2**	68.2	60.8	12.5	–	11.2	6.3	26.9	72.6
			Student ($\nu = 6$)	75.6	68.9	84.6	73.1	69.2	27.1	–	12.6	7.9	**94.3**	79.5
			Clayton	2.6	15.5	83.6	94.6	**97.7**	36.5	–	22.7	8.6	79.5	97.4
			Gumbel	20.3	5.0	35.7	22.2	63.2	**87.7**	–	39.8	7.8	43.7	60.4
			Frank	*4.5*	*5.1*	*4.7*	*5.2*	*4.8*	*4.8*	–	*5.5*	*5.1*	*4.9*	*4.8*
		0.4	Gaussian	11.7	62.0	**93.6**	81.4	60.1	24.2	–	25.7	8.2	78.1	73.4
			Student ($\nu = 6$)	47.8	55.9	**95.2**	91.3	74.1	38.4	–	28.3	10.8	90.9	86.2
			Clayton	1.3	18.1	98.7	99.8	**99.9**	69.4	–	81.0	39.4	98.5	**99.9**
			Gumbel	26.5	7.9	72.8	29.5	74.7	**93.7**	–	50.3	11.0	67.6	77.0
			Frank	*5.0*	*4.8*	*4.6*	*5.2*	*5.1*	*5.5*	–	*4.7*	*4.4*	*4.9*	*5.0*
	500	0.2	Gaussian	47.7	94.1	**100**	**100**	99.8	99.0	–	66.6	15.1	**100**	**100**
			Student ($\nu = 6$)	**100**	**100**	**100**	**100**	**100**	**100**	–	77.4	32.3	**100**	**100**
			Clayton	6.3	82.8	**100**	**100**	**100**	**100**	–	93.7	35.8	**100**	**100**
			Gumbel	71.4	6.0	95.9	74.3	**100**	**100**	–	98.5	34.1	98.9	**100**
			Frank	*4.5*	*4.8*	*4.3*	*5.1*	*5.2*	*5.3*	–	*5.6*	*5.3*	*5.5*	*5.1*
		0.4	Gaussian	**100**	**100**	**100**	**100**	99.9	93.1	–	97.6	37.9	**100**	**100**
			Student ($\nu = 6$)	**100**	**100**	**100**	**100**	**100**	99.7	–	98.6	61.5	**100**	**100**
			Clayton	8.3	83.7	**100**	**100**	**100**	**100**	–	**100**	99.6	**100**	**100**
			Gumbel	93.3	16.3	**100**	95.1	**100**	**100**	–	99.9	62.5	**100**	**100**
			Frank	*5.0*	*4.6*	*4.7*	*4.9*	*4.6*	*4.2*	–	*5.3*	*4.7*	*4.4*	*4.6*

Note: Numbers in *italics* are nominal levels and should correspond to the size of 5%. Numbers in **bold** indicate the best performing approach.