An Introduction to Copulae

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Outline

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4. Copula Families
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1. Introduction

- Dependency modelling
- Linear correlation coefficient - a measure of linear dependence
- In e.g. financial markets we often see non-linear dependency structures
- Elliptical distributions - linear dependence structure - correlation coefficient meaningful
- Non-elliptical distributions - alternative measures of dependence needed ⇒ Copulae
- Any multivariate distribution function can serve as a copula
1.1. Brief historical background:

- 1940’s: Hoeffding studies properties of multivariate distributions
- 1959: The word **copula** appears for the first time (Sklar)
- 1999: Introduced to financial applications (Embrechts, McNeil, Straumann)
- 2006: Several insurance companies, banks and other financial institutions apply copulæ as a risk management tool
2. Definitions and Theorems

**Definition (Copula)**

A \textit{d-dimensional copula} is a multivariate distribution, \( C \), with standard uniform marginal distributions.

**Theorem (Sklar)**

Every multivariate distribution \( F \), with margins, \( F_1, F_2, \ldots, F_d \) can be written as

\[
F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)),
\]

for some copula \( C \).
2. Definitions and Theorems

- Given a random vector $\mathbf{X} = (X_1, \ldots, X_d)$ the copula of their joint distribution function may be extracted from equation (2.1):

$$C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)),$$

where the $F_i^{-1}$'s are the quantile functions of the margins.

- The copula is often represented by its density function $c(u)$:

$$C(u) = P(U_1 \leq u_1, U_2 \leq u_2, \ldots, U_d \leq u_d) = \int_0^{u_1} \ldots \int_0^{u_d} c(u) \, du,$$
2. Definitions and Theorems

- For the implicit copula of an absolutely continuous joint df $F$ with strictly continuous marginal df’s $F_1, \ldots, F_d$, the copula density is given by

$$c(u) = \frac{f(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))}{f_1(F_1^{-1}(u_1)) \cdots f_d(F_d^{-1}(u_1))}.$$ 

- Hence,

$$c(F_1(x_1), \ldots, F_d(x_d)) = \frac{h(x_1, \ldots, x_d)}{f_1(x_1) \cdots f_d(x_d)}.$$ 

- This means that a general $d$-dimensional density can be written as

$$f(x_1, \ldots, x_d) = c(F_1(x_1), \ldots, F_d(x_d)) \cdot f_1(x_1) \cdots f_d(x_d)$$

for some copula density $c(\cdot)$. 

2.1. Attractive features of copulae:

- A copula describes how the marginals are tied together in the joint distribution.
- The joint df is decomposed into the marginal dfs and a copula.
- The marginal dfs and the copula can be modelled and estimated separately, independent of each other.
- Given a copula, we can obtain many multivariate distributions by selecting different marginal dfs.
- The copula is invariant under increasing and continuous transformations.
2.2. Examples

Example 1: Independence copula

If $U \sim U(0, 1)$ and $V \sim U(0, 1)$ are independent, then

$$C(u, v) = uv = \prod = P(U \leq u)P(V \leq v) = P(U \leq u, V \leq v) = H(u, v),$$

where $H(u, v)$ is the distribution function of $(U, V)$. $C$ is called the independence copula.

Figure: Simulations from the bivariate independence copula.
2.2 Examples

Example 2: Gaussian copula (implicit)

\[ C_{\rho}^{Ga}(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi(1 - \rho^2)^{1/2}} \exp \left\{ -\frac{x^2 - 2\rho xy + y^2}{2(1 - \rho^2)} \right\} \, dx \, dy, \]

where \( \rho \) is the linear correlation coefficient.

Example 3: Student’s t copula (implicit)

\[ C_{\rho, \nu}^{t}(u, v) = \int_{-\infty}^{t^{-1}(u)} \int_{-\infty}^{t^{-1}(v)} \frac{1}{2\pi(1 - \rho^2)^{1/2}} \left\{ 1 + \frac{x^2 - 2\rho xy + y^2}{\nu(1 - \rho^2)} \right\}^{-\nu/2} \, dx \, dy, \]

where \( \nu \) is the degrees of freedom and \( \rho \) is the linear correlation coefficient.
2.2 Examples 2-3: Illustration

Figure: Simulations from the bivariate Gaussian- and Student’s t distribution, and the associated copulae ($\rho = 0.7$, $\nu = 4$).
2.2 Examples

Example 4: Clayton copula (explicit)

\[ C^C_l(\delta; u, v) = (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta} , \]

where \( 0 < \delta < \infty \) is the parameter controlling the dependence. Perfect dependence is obtained if \( \delta \to \infty \), while \( \delta \to 0 \) implies independence.

Figure: Simulations from the bivariate Clayton copula (\( \delta = 3 \)).
2.2 Examples

Example 5: Gumbel copula (explicit)

\[ C^G_u(u, v) = \exp\{-[(-\ln u)^\theta + (-\ln v)^\theta]^{1/\theta}\}, \]

where \( 1 < \theta < \infty \) is the parameter controlling the dependence. Perfect dependence is obtained if \( \theta \to \infty \), while \( \theta \to 1 \) implies independence.

Figure: Simulations from the bivariate Gumbel copula (\( \theta = 3 \)).
3. Dependence Concepts

We will consider the following dependence measures:

▷ Linear correlation

▷ Concordance
  ○ Kendall’s tau
  ○ Spearman’s rho

▷ Tail dependence
3.1. Linear correlation

\[ \rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}. \]

- Sensitive to outliers
- Measures the "average dependence" between \( X \) and \( Y \)
- Invariant under strictly increasing \textbf{linear} transformations
- May be misleading in situations where multivariate df is not elliptical
3.1. Linear correlation

Figure: Illustration of the potential pitfalls of the linear correlation coefficient. Both distributions have linear correlation coefficient equal to 0.7.
3.2. Concordance

Let \((x_i, y_i)\) and \((x_j, y_j)\) be two observations from a random vector \((X, Y)\) of continuous random variables.

- **Concordance:** \((x_i - x_j)(y_i - y_j) > 0\)
- **Discordance:** \((x_i - x_j)(y_i - y_j) < 0\)

Let \((X_1, Y_1)\) and \((X_2, Y_2)\) be independent vectors of cont. random variables with joint df’s \(H_1\) and \(H_2\) and copulae \(C_1\) and \(C_2\), respectively. Let \(Q\) define the difference between the prob. of concordance and discordance of \((X_1, Y_1)\) and \((X_2, Y_2)\):

\[
Q = \mathbb{P} ((X_1 - X_2)(Y_1 - Y_2) > 0) - \mathbb{P} ((X_1 - X_2)(Y_1 - Y_2) < 0)
\]

\[
= Q(C_1, C_2) = 4 \int_0^1 \int_0^1 C_2(u, v)dC_1(u, v) - 1.
\]
3.2.1. Kendall’s tau

$$\rho_\tau(X, Y) = Q(C, C) = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1$$

$$= 4 \mathbb{E}(C(U, V)) - 1.$$

- Less sensitive to outliers
- Measures the "average dependence" between $X$ and $Y$
- Invariant under strictly increasing transformations
- Depends only on the copula of $(X, Y)$
- For elliptical copulae: $\text{cor}(X, Y) = \sin \left( \frac{\pi}{2} \rho_\tau \right)$
3.2.2. Spearman's rho

\[ \rho_S(X, Y) = 3Q(C, \Pi) \]

\[ = 12 \int_0^1 \int_0^1 uvdC(u, v) - 3 \]

\[ = 12 \int_0^1 \int_0^1 C(u, v)dudv - 3. \]

▷ Less sensitive to outliers
▷ Measures the "average dependence" between \( X \) and \( Y \)
▷ Invariant under strictly increasing transformations
▷ Depends only on the copula of \( (X, Y) \)
▷ \( \rho_S(X, Y) = \rho(F_X(X), F_Y(Y)) \)
▷ For elliptical copulae: \( \text{cor}(X, Y) = 2\sin \left( \frac{\pi}{6} \rho_S \right) \)
3.3. Tail dependence

Let \((X, Y)\) be a r.v. with marginal df’s \(F_X\) and \(F_Y\). The coefficient of upper and lower tail dependence of \((X, Y)\) is defined as:

\[
\lambda_u(X, Y) = \lim_{\alpha \to 1} \mathbb{P}(Y > F_Y^{-1}(\alpha) | X > F_X^{-1}(\alpha)),
\]

\[
\lambda_l(X, Y) = \lim_{\alpha \to 0} \mathbb{P}(Y \leq F_Y^{-1}(\alpha) | X \leq F_X^{-1}(\alpha)).
\]

I.e. the tail dependence is the prob. of observing a large(small) \(Y\), given that \(X\) is large(small). If \(\lambda_u > 0\) (\(\lambda_l > 0\)), then we say that \((X, Y)\) has upper (lower) tail dependence.

- Gaussian copula: \(\lambda_u = \lambda_l = 2 \lim_{x \to \infty} \Phi \left( x \sqrt{1 - \rho / \sqrt{1 + \rho}} \right) = 0\)

- Student-t copula: \(\lambda_u = \lambda_l = 2t_{\nu+1} \left( -\sqrt{\nu + 1} \sqrt{(1 - \rho) / (1 + \rho)} \right)\). Asymptotic tail dependence, even when \(\rho = 0\).

- Clayton copula: \(\lambda_u = 0, \lambda_l = 2^{-1/\delta}\).

- Gumbel copula: \(\lambda_l = 0, \lambda_u = 2 - 2^{1/\theta}\).
4. Copula Families

We will consider the two most important families of copulae:

- Elliptical copulae
- Archimedean copulae
4.1. Elliptical Copulae

- Implied by well-known multivariate df’s, derived through Sklar’s theorem
- Extends the multivariate normal $\mathcal{N}_d(\mu, \Sigma)$.
- Extend to arbitrary dimensions and are rich in parameters. A $d$-dim elliptical copula has at least $d(d - 1)/2$ parameters
- Easy to simulate
- Drawback: Do not have closed form expressions and are restricted to have radial symmetry

**Examples**: Gaussian copula, Student’s t copula
4.2. Archimedean Copulae

An Archimedean copula is defined as follows:

\[ C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v)). \]

The function \( \varphi \) is called the generator of the copula.

- Allow for a great variety of dependence structures
- Closed form expressions
- \textbf{Not} derived from mv df’s using Sklar’s theorem
- Drawback: Higher dimensional extensions difficult

\textbf{Examples:} Clayton copula, Gumbel copula
4.2. Archimedean Copulae

**Example 1: Clayton copula**
The generator function for the Clayton copula is given by $\varphi(t) = (t^{-\delta} - 1)/\delta$, where $\delta \in (0, \infty)$. This gives the Clayton copula:

$$C_\delta(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v)) = (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta}.$$  

The Clayton copula has lower tail dependence.

**Example 2: Gumbel copula**
The generator function for the Gumbel copula is given by $\varphi(t) = (-\ln t)^\theta$, where $\theta \geq 1$. This gives the Gumbel copula:

$$C_\theta(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v)) = \exp(-[(\ln u)^\theta + (\ln v)^\theta]^{1/\theta}).$$

The Gumbel copula has upper tail dependence.
5. Estimating Copula Parameters

**Fully parametric method:**
- Denoted Inference functions for margins (IFM) method.
- Assumes parametric univariate marginal distributions.
- Parameters of margins are first estimated, then each parametric margin is plugged into the copula likelihood, and this full likelihood is maximized.
- Success depends upon finding appropriate parametric models for the margins, which is not always straightforward

**Semi-parametric method:**
- Denoted the pseudo-likelihood or canonical maximum likelihood (CML) method
- No parametric assumptions for the margins, use empirical cdf’s, then plug into likelihood
5.1 Estimation - Elliptical copulae

Gaussian copula:
- Correlation matrix \( R \) \((d(d - 1)/2 \) parameters)\)
- ML estimator: \( \hat{R} = \arg \max_{R \in \mathcal{P}} \sum_{j=1}^{n} \log c(U_j; R) \), where the pseudo samples \( U_j \) are generated using either the IFM or the CML method.

Student’s t copula:
- Correlation matrix \( R \) and degree-of-freedom \( \nu \) \((1 + d(d - 1)/2 \) parameters)\)
- ML wrt \( R \) and \( \nu \) simultaneously difficult
- Simpler: two-stage approach in which \( R \) is estimated first using Kendall’s tau, and then the pseudo-likelihood function is maximized wrt \( \nu \).
5.2 Estimation - Archimedean copulae

Clayton and Gumbel copulae:

- One parameter, $\delta$ and $\theta$ respectively
- Numerical optimization of likelihood
- Bivariate - utilize the following relationships to Kendall’s tau:

$$\hat{\delta} = \frac{2\hat{\rho}_\tau}{1 - \hat{\rho}_\tau}, \quad \hat{\theta} = \frac{1}{1 - \hat{\rho}_\tau}.$$
6. Simulating from Copulae

**Gaussian copula:**
- Simulate \( X \sim \mathcal{N}_d(0, R) \)
- Set \( U = (\Phi(X_1), \ldots, \Phi(X_d)) \) or \( U = (F(X_1), \ldots, F(X_d)) \) where the \( F \)'s are the quantile functions

**Student’s t copula:**
- Simulate \( X \sim t_d(0, R, \nu) \)
- Set \( U = (t_\nu(X_1), \ldots, t_\nu(X_d)) \) or \( U = (F(X_1), \ldots, F(X_d)) \) where the \( F \)'s are the quantile functions
6. Simulating from Copulae

**Clayton copula:**
By noting that the inverse of the generator is equal to the Laplace transform of a Gamma variate \( X \sim Ga(1/\delta, 1) \), the simulation algorithm becomes:

- Simulate a gamma variate \( X \sim Ga(1/\delta, 1) \)
- Simulate \( d \) iid \( U(0, 1) \) variables \( V_1, \ldots, V_d \)
- Return \( U = \left( (1 - \frac{\log V_1}{X})^{-1/\delta}, \ldots, (1 - \frac{\log V_d}{X})^{-1/\delta} \right) \)

**Gumbel copula:**
By noting that the inverse of the generator function is equal to the Laplace transform of a positive stable variate \( X \sim St(1/\theta, 1, \gamma, 0) \), where \( \gamma = (\cos \left( \frac{\pi}{2\theta} \right))^\theta \) and \( \theta > 1 \), the simulation algorithm becomes:

- Simulate a positive stable variate \( X \sim St(1/\theta, 1, \gamma, 0) \)
- Simulate \( d \) iid \( U(0, 1) \) variables \( V_1, \ldots, V_d \)
- Return \( U = \left( \exp \left( - \left( - \frac{\log V_1}{X} \right)^{1/\theta} \right), \ldots, \exp \left( - \left( - \frac{\log V_d}{X} \right)^{1/\theta} \right) \right) \)
6. Simulating from Copulae

In general we could apply the conditional marginal cdf's:

\[
F_{i|1,\ldots,i-1}(u_i|u_1, \ldots, u_{i-1}) = \frac{\partial^{i-1} C(u_1, \ldots, u_i)}{\partial u_1 \cdots \partial u_{i-1}} \frac{\partial^{i-1} C(u_1, \ldots, u_{i-1})}{\partial u_1 \cdots \partial u_{i-1}}.
\]

The simulation algorithm then becomes:

- Simulate a rv \( u_1 \) from \( U(0, 1) \),
- Simulate a rv \( u_2 \) from \( F_{2|1}(\cdot|u_1) \),
- \[ \vdots \]
- Simulate a rv \( u_d \) from \( F_{d|1,\ldots,d-1}(\cdot|u_1, \ldots, u_{d-1}) \).
- Generally means simulating a rv \( V_i \) from \( U(0, 1) \) from which \( u_i = F_{i|1,\ldots,i-1}^{-1}(V_i|u_1, \ldots, u_{i-1}) \) can be obtained, if necessary by numerical root finding.
7. Higher Dimensional Copulae

I. Copulae with at least $d(d - 1)/2$ bivariate dependence parameters:
   - Build multivariate copulae from bivariate copula
   - Based on iteratively mixing conditional copulae
   - Very flexible tool for dependency modelling
   - Does not require any assumption of conditional independence
   - Also referred to as ‘Vines’ (Cooke and Bedford, 2002)
   - Drawback: difficult, slow, depends heavily on permutation
7. Higher Dimensional Copulae

I. Copulae with at least $d(d - 1)/2$ bivariate dependence parameters:

Example:

$$c_{1234}(u_1, u_2, u_3, u_4) = c_{12}(u_1, u_2) \cdot c_{23}(u_2, u_3) \cdot c_{34}(u_3, u_4)$$

$$\cdot c_{13|2}(F_{1|2}(u_1 | u_2), F_{3|2}(u_3 | u_2)) \cdot c_{24|3}(F_{2|3}(u_2 | u_3), F_{4|3}(u_4 | u_3))$$

$$\cdot c_{14|23}(F_{1|23}(u_1 | u_2, u_3), F_{4|23}(u_4 | u_2, u_3)),$$

where $F_{i|1\ldots i-1}(u_j | u_1, \ldots, u_{i-1}) = \frac{\partial^{i-1} C(u_1, \ldots, u_i)}{\partial u_1 \ldots \partial u_{i-1}} \Bigg/ \frac{\partial^{i-1} C(u_1, \ldots, u_{i-1})}{\partial u_1 \ldots \partial u_{i-1}}$. 

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Introduction to Copulae
7. Higher Dimensional Copulae

II. Archimedean copulae with \( d - 1 \) bivariate dependence parameters:

- Build multivariate copulae from bivariate copula
- Based on iteratively mixing conditional copulae
- Less flexible but more intuitive and faster than ’vines’
- Only applicable to Archimedean copulae with strict generator functions

\[
C^3(u_1, u_2, u_3) = \varphi^{-1}[\varphi(u_1) + \varphi(u_2) + \varphi(u_d)]
\]
\[
= \varphi^{-1}[\varphi^{-1}[\varphi(u_1) + \varphi(u_2)]] + \varphi(u_3)
\]
\[
= C^2(C^2(u_1, u_2), u_3),
\]
\[
\Rightarrow C^d(u_1, \ldots, u_d) = C^2(C^{d-1}(u_1, \ldots, u_{d-1}), u_d).
\]

Example:

\[
C^3(u_1, u_2, u_3) = \varphi^{-1}_2 \circ \varphi^{-1}_1[\varphi_1(u_1) + \varphi_1(u_2)] + \varphi_2(u_3)].
\]
8. Application

Using some simplifying assumptions we simulated losses for a portfolio of 30 American firms, assuming Gaussian, Student’s t and Vine (Student’s t C-vine) dependency structures.

Figure: Value-at-risk for a portfolio of 30 American firms, assuming Gaussian, Student’s t and Vine dependency structure.
9. Copula Goodness-of-fit Testing

- Special case of testing for multivariate density models
- Complicated due to the unspecified marginal df’s. Asymptotic distributional properties becomes very difficult to derive ⇒ $p$-values obtained through simulation.
- $\chi^2$- and other tests based on binning the probability space will not be feasible in higher dimensions as the need for data would be too great.
- Some tests focus on multivariate smoothing procedures. These are computationally very demanding in high dimensions.
- A more promising class of tests project the multivariate problem to a univariate problem, then apply a univariate GOF statistic, e.g. Anderson-Darling (AD).
- We may base the testing on the probability integral transform (PIT).
9.1 Probability Integral Transform (PIT)

- The PIT transforms a set of dependent variables into a new set of independent $U(0, 1)$ variables, given the multivariate distribution.
- A universally applicable way of creating a set of iid $U(0, 1)$ variables from any data set with known distribution
- First introduced by Rosenblatt (1952)
- Inverse of simulation
- GOF: the observed copula is PIT assuming a $\mathcal{H}_0$ copula. Then a test of independence is performed.
9.1 Probability Integral Transform (PIT)

**DEFINITION: Probability Integral Transform**

Let $X = (X_1, \ldots, X_d)$ denote a random vector with marginal distributions $F_i(x_i) = P(X_i \leq x_i)$ and conditional distributions $F(X_i \leq x_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1})$ for $i = 1, \ldots, d$. The PIT of $X$ is defined as $T(X) = (T_1(X_1), \ldots, T_d(X_d))$ where $T_i(X_i)$ is defined as follows:

- $T_1(X_1) = P(X_1 \leq x_1) = F_{X_1}(x_1)$,
- $T_2(X_2) = P(X_2 \leq x_2 | X_1 = x_1) = F_{X_2|X_1}(x_2 | x_1)$,
- $\vdots$
- $T_d(X_d) = P(X_d \leq x_d | X_1 = x_1, \ldots, X_{d-1} = x_{d-1}) = F_{X_d|X_1\ldots X_{d-1}}(x_d | x_1, \ldots, x_{d-1})$.

The random variables $Z_i = T_i(X_i)$, for $i = 1, \ldots, d$ are uniformly and independently distributed on $[0, 1]^d$. $F(x_i | x_1, \ldots, x_{i-1})$ is found by

$$F_{i|1\ldots i-1}(u_i | u_1, \ldots, u_{i-1}) = \frac{\partial^{i-1} C(u_1, \ldots, u_i)}{\partial u_1 \cdots \partial u_{i-1}} \bigg/ \frac{\partial^{i-1} C(u_1, \ldots, u_{i-1})}{\partial u_1 \cdots \partial u_{i-1}}.$$
9.2 Proposed tests

$G$: Breymann et al. (2003)

\[ Y_j^G = \sum_{i=1}^{d} \Phi^{-1}(z_{ji})^2, \quad j = 1, \ldots, n, \]

\[ G(w) = P \left( \chi_{d}^2 \left( Y_j^G \leq w \right) \right), \quad w \in [0, 1]. \]

- Coincides with the tests proposed by Malevergne and Sornette (2003) when the latter is based on PIT. Also coincides with the test proposed by Chen et al. (2004).
- Very fast
- Tail weight
- **NOT** consistent
9.2 Proposed tests

B: Berg and Bakken (2005)

\[ z_{ji}^* = P(r_i \leq \tilde{z}_{ji} | r_1, \ldots, r_{i-1}) = \left(1 - \left(\frac{1 - \tilde{z}_{ji}}{1 - r_{i-1}}\right)\right)^{d-(i-1)}, \]

\[ Y_j^B = \sum_{i=1}^{d} \gamma(z_{ji}; \alpha) \cdot \Phi^{-1}(z_{ji}^*)^2, \quad j = 1, \ldots, n, \]

where \( \gamma(\cdot) \) is a weight function and \( \alpha \) are weight parameters. Then

\[ B(w) = P(F_B(Y^B) \leq w), \quad w \in [0, 1]. \]

- Similar to G-test but based on transformed data \( Z^* \).
- Fast
- Any weight
- Consistent
9.2 Proposed tests

**Q: Panchenko (2005)**

\[
Q = \langle f_1 - f_2 | \kappa_d | f_1 - f_2 \rangle = Q_{11} - 2Q_{12} + Q_{22},
\]

\[
\hat{Q}_{kp} = \frac{1}{n^2} \sum_{j=1}^{n} \sum_{p=1}^{n} \kappa_d(x_k^{jp}, x_p^{jp}),
\]

where

\[
\kappa_d(x_1, x_2) = \exp \left\{ -\|x_1 - x_2\|^2 / (2dh^2) \right\}.
\]

- Based on positive bilinear forms
- Very slow
- No weight
- Consistent
9.2 Proposed tests

**K: Genest et al. (2006)**

\[
K(w) = P(C(Z) \leq w), \quad w \in [0, 1],
\]

\[
\hat{K}(w) = \frac{1}{n+1} \sum_{j=1}^{n} I(\hat{C}(z_j) \leq w), \quad w = \frac{1}{n+1}, \ldots, \frac{n}{n+1}.
\]

- Based on the empirical copula and Kendall’s process
- Slow
- Left tail weight
- Consistent
9.3 Mixing results

Mix a Gaussian copula with an alternative copula to construct a mixed copula

\[ C_{\text{mix}} = (1 - \beta) \cdot C_{\text{Ga}} + \beta \cdot C_{\text{Alt}}, \quad \beta \in [0, 1] \]

- \( C_{\text{Alt}} \): Student’s t (\( C_{\text{St}}^{\nu} \), \( \nu = 4 \)), Clayton (\( C_{\text{Cl}}^{\delta} \), \( \delta = 1.0 \)) and survival Clayton (\( C_{\text{sCl}}^{\delta} \), \( \delta = 1.0 \))

- \( H_0 \): \( C_{\text{Ga}} \)

Simulate from \( C_{\text{Ga}} \) and \( C_{\text{Alt}} \) and mix. Then PIT \( C_{\text{mix}} \) under \( H_0 \). Finally compute test statistic and corresponding \( p \)-value

Repeat 500 times to obtain rejection rates

Consider G, Q, K and B test. For the B-test we consider no weight and power tail weighting:

\[ \gamma(Z_i; \alpha) = (Z_i - \frac{1}{2})^{\alpha}, \quad \alpha = [4, 10]. \]
9.3 Mixing results

![Graph showing mixing results with different copula families and parameters]

\[ C^{\text{Alt}} = C^{\text{St}}, \ d=2, \ n=500, \ \nu=4 \]

- G
- Q
- K
- B
- B(\alpha=4)
- B(\alpha=10)
9.3 Mixing results

\[ C^{\text{Alt}} = C^{\text{St}}, \ d=2, \ n=2500, \ \nu=4 \]
9.3 Mixing results

\[ C^{\text{Alt}} = C^{\text{St}}, \ d=5, \ n=500, \ \nu=4 \]

- G
- Q
- K
- B
- B(\alpha=4)
- B(\alpha=10)
9.3 Mixing results

\[ C^{\text{Alt}} = C^{\text{St}}, \ d=5, \ n=2500, \ \nu=4 \]
9.3 Mixing results

\[ C^{\text{Alt}} = C^\text{St}, \quad d=10, \quad n=2500, \quad \nu=4 \]
9.3 Mixing results

\[ C_{\text{Alt}} = C_{\text{Cl}}, \; d=2, \; n=500, \; \delta=1 \]

\[
\begin{array}{c}
\text{Rejection Rate} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\(\beta\)} \\
0.0 & 0.2 & 0.4 & 0.6 & 0.8 & 1.0 \\
\end{array}
\]
9.3 Mixing results

$C^{\text{Alt}} = C^{\text{Cl}}, d=2, n=2500, \delta=1$
9.3 Mixing results

\[ C^{\text{Alt}} = C^{\text{CI}}, \, d=5, \, n=500, \, \delta=1 \]
9.3 Mixing results

\[ C^{\text{Alt}} = C^{\text{Cl}}, \, d=5, \, n=2500, \, \delta=1 \]
9.3 Mixing results

\[ C^{\text{Alt}} = C^{\text{Cl}}, \quad d=10, \quad n=2500, \quad \delta=1 \]
9.3 Mixing results

\[ C^{\text{Alt}} = C^{s\text{CI}}, \ d=2, \ n=500, \ \delta=1 \]
9.3 Mixing results

$C_{\text{Alt}} = C_{\text{sCl}}, \ d=2, \ n=2500, \ \delta=1$

- $G$
- $Q$
- $K$
- $B$
- $B(\alpha=4)$
- $B(\alpha=10)$
9.3 Mixing results

\[ C^{\text{Alt}} = C^{sCl}, \quad d=5, \quad n=500, \quad \delta=1 \]

Graph showing the rejection rate for different copula families and parameter values. The graph includes lines for different copula families such as G, Q, K, B, B(\(\alpha=4\)), and B(\(\alpha=10\)). The x-axis represents \(\beta\), and the y-axis represents the rejection rate.
9.3 Mixing results

\[ C^{\text{Alt}} = C^{s\text{Cl}}, \ d=5, \ n=2500, \ \delta=1 \]

- G
- Q
- K
- B
- B(\alpha=4)
- B(\alpha=10)
9.3 Mixing results

\[ C_{\text{Alt}}^{sCl} = C^{sCl}, \ d=10, \ n=2500, \ \delta=1 \]

- **G**
- **Q**
- **K**
- **B**
- \( B(\alpha=4) \)
- \( B(\alpha=10) \)
10. Summary

- Linear correlation coefficient not sufficient outside the world of elliptical distributions ⇒ alternative dependence measures
- Copula families: Elliptical, Archimedean
- Estimation and simulation
- Complex multivariate highly dependent models can be built, based on bivariate copulae
- Significant impacts, i.e. on portfolio VaR
- Goodness-of-fit:
  - Bivariate: several candidates
  - Dimension > 2: B-test
References


